SOME THEOREMS ON THE CONVERGENCE OF SERIES.

BY PROFESSOR R. D. CARMICHAEL.

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No general criteria are known for the study of the convergence of a series of complex terms* when the series does not converge absolutely. The object of this note is to generalize one of the few important special methods which are already available for such study. The theorems obtained are of general applicability; but their practical value lies chiefly in their use in investigating the convergence of series which do not converge absolutely.

In § 1 I give two lemmas which are valuable in themselves; the second is fundamental in the present paper. In § 2 are two general theorems concerning the convergence of series of constant terms, where the convergence need not be absolute. They are generalizations of important results due to Dedekind. In § 3 two corresponding theorems for uniform convergence are given.

§ 1. *Statement and Proof of Two Lemmas.*—It is convenient to begin with the proof of two lemmas.

LEMMA I. *Let* $s_1, s_2, s_3, \cdots$ *be any infinite sequence of numbers having the finite limit* s. *Then*

$$\lim_{n=\infty} \frac{1}{n^{k+1}} \sum_{i=1}^{n} i^k s_i = \frac{s}{k+1},$$

k being a positive integer or zero.†

If ϵ is any positive number there exists a positive integer t such that $|s_i - s| < \epsilon/2$ for every value of i greater than t. For $n > t$, let us write

$$\frac{1}{n^{k+1}} \sum_{i=1}^{n} i^k s_i = \frac{1}{n} \sum_{i=1}^{t} \frac{i^k}{n^k} s_i + \frac{1}{n} \sum_{i=t+1}^{n} \frac{i^k}{n^k} s + \frac{1}{n} \sum_{i=t+1}^{n} \frac{i^k}{n^k} (s_i - s).$$

* The same statement is also true of the special case where the terms are real and are alternately positive and negative.

† For the case when $k = 0$ the lemma has been proved by Cesàro, *Darboux's Bulletin* (2), 14 (1890), pp. 114–120. See Whittaker, Modern Analysis, p. 26.

Then we have

$$\left|\frac{1}{n^{k+1}}\sum_{i=1}^{n} i^k s_i - \frac{1}{n}\sum_{i=t+1}^{n}\frac{i^k}{n^k}s\right| \leqq \frac{1}{n}\sum_{i=1}^{t}|s_i| + \frac{1}{n}\sum_{i=t+1}^{n}|s_i - s|$$

$$< \frac{tA}{n} + \frac{n-t}{n}\frac{\epsilon}{2},$$

where A is a positive constant such that $|s_i| < A$ for every i. It is clear that an N exists such that the last member is less than ϵ for every n greater than N. Therefore it follows at once that we have

$$\lim_{n=\infty}\left[\frac{1}{n^{k+1}}\sum_{i=1}^{[n}} i^k s_i - \frac{1}{n}\sum_{i=t+1}^{n}\frac{i^k}{n^k}s\right] = 0.$$

Now,

$$1^k + 2^k + 3^k + \cdots + n^k = \frac{n^{k+1}}{k+1} + \frac{1}{2}n^k + \cdots,$$

the second member being a polynomial in n of degree $k+1$; hence we see readily that

$$\lim_{n=\infty}\frac{1}{n}\sum_{i=t+1}^{n}\frac{i^k}{n^k}s = s\lim_{n=\infty}\frac{(t+1)^k + (t+2)^k + \cdots + n^k}{n^{k+1}} = \frac{s}{k+1}.$$

From this and the preceding limit follows at once the limit given in the conclusion of the lemma.

COROLLARY. *Let $s_1, s_2, s_3, \cdots$ be an infinite sequence of functions of the complex variable x regular in a given closed domain D and converging uniformly to the limit function s. Then the limit*

$$\lim_{n=\infty}\frac{1}{n^{k+1}}\sum_{i=1}^{n} i^k s_i$$

exists uniformly in D and is $s/(k+1)$.

Certain obvious verbal changes in the proof of the lemma lead readily to the proof of the corollary.

LEMMA II. *Let $s_1, s_2, s_3, \cdots$ be any infinite sequence of numbers having the finite limit s. Then every solution u_n of the difference equation**

$$\Delta^k u_n = s_n$$

* We suppose here that a solution is defined only for non-negative integral values of the argument n.

has the property that

$$\lim_{n=\infty} \frac{u_n}{n^k} = \frac{s}{k!}.$$

If $\bar{u}_n$ is any particular solution of the equation, then the general solution is

$$u_n = \alpha_0 + \alpha_1 n + \alpha_2 n^2 + \cdots + \alpha_{k-1} n^{k-1} + \bar{u}_n,$$

where $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_{k-1}$ are independent of n. Hence, if the conclusion of the theorem is valid for any particular solution $\bar{u}_n$ of the equation, it is valid for every solution. Accordingly we shall prove it for the particular solution defined thus: Write

$$\Delta^{k-1}\bar{u}_n = \sum_{i=1}^{n} s_i \equiv n s_n^{(1)} \qquad (n > 0).$$

Then, by Lemma I, we see that $\lim_{n=\infty} s_n^{(1)} = s$. Then write

$$\Delta^{k-2}\bar{u}_n = \sum_{i=1}^{n} i s_i^{(1)} \equiv n^2 s_n^{(2)} \qquad (n > 0).$$

We have $\lim_{n=\infty} s_n^{(2)} = s/2!$, as we see again by aid of Lemma I. By continuing thus, the proof of the lemma is readily completed.

COROLLARY. *Let $s_1, s_2, s_3, \cdots$ be an infinite sequence of functions of the complex variable x regular in a given closed domain D and converging uniformly to the limit function s. Then every solution u_n of the difference equation*

$$\Delta^k u_n = s_n$$

which is regular throughout D has the property that

$$\lim_{n=\infty} \frac{u_n}{n^k}$$

exists uniformly in D and is $s/k!$.

The modifications of the preceding argument, which it is necessary to make to establish the corollary, are obvious.

§ 2. *Two Theorems on the Convergence of Series of Constant*

Terms.—We are now in position to prove the following theorem:

THEOREM I. *Let $a_1, a_2, a_3, \cdots$ be any infinite sequence of numbers such that each of the limits**

$$(1)\quad \lim_{n=\infty} S_n^{(k)},\ \lim_{n=\infty} nS_n^{(k-1)},\ \lim_{n=\infty} n^2S_n^{(k-2)},\ \cdots,\ \lim_{n=\infty} n^{k-1}S_n^{(1)}$$

exists and is finite, where

$$S_n^{(1)} = a_1 + a_2 + \cdots + a_n,$$

$$S_n^{(2)} = S_1^{(1)} + S_2^{(1)} + \cdots + S_n^{(1)},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$S_n^{(k)} = S_1^{(k-1)} + S_2^{(k-1)} + \cdots + S_n^{(k-1)}.$$

Let $c_1, c_2, c_3, \cdots$ be an infinite sequence of numbers such that

$$(2)\quad \sum_{i=1}^{\infty} |\Delta^k c_i|$$

is convergent.

Then the series

$$a_1c_1 + a_2c_2 + a_3c_3 + \cdots$$

is convergent (but not necessarily absolutely convergent).

It is easy to construct an example to show that the last series is not always absolutely convergent. Hence we have to prove only the affirmative statement in the conclusion of the theorem.

If we adopt the convention that a letter with a negative or a zero subscript has the value zero, we may write†

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* In case $k = 1$ only the first of these limits is to be taken into consideration. The condition on the a's then is obviously the condition that the series $a_1 + a_2 + a_3 + \cdots$ converges. For this case (namely, when $k = 1$) the theorem is due to Dedekind (Dirichlet-Dedekind, Zahlentheorie, second edition, 1871, p. 373). It was rediscovered by Jensen.

† Here we employ repeatedly a fundamental identity due to Abel (*Crelle*, vol. 1 (1826), p. 314; *Oeuvres*, 1, p. 222), namely,

$$\sum_{i=1}^{n} v_iu_i \equiv \sum_{i=1}^{n} (V_i - V_{i-1})u_i \equiv \sum_{i=1}^{n} V_i(u_i - u_{i+1}) + V_nu_{n+1},$$

where

$$V_i = v_1 + v_2 + \cdots + v_i,\quad i \geqq 1,\quad V_0 = 0.$$

$$S_n \equiv \sum_{i=1}^{n} a_i c_i = \sum_{i=1}^{n} (S_i^{(1)} - S_{i-1}^{(1)}) c_i$$

$$= -\sum_{i=1}^{n} S_i^{(1)} \Delta c_i + S_n^{(1)} c_{n+1}$$

$$= -\sum_{i=1}^{n} (S_i^{(2)} - S_{i-1}^{(2)}) \Delta c_i + S_n^{(1)} c_{n+1}$$

$$= \sum_{i=1}^{n} S_i^{(2)} \Delta^2 c_i - S_n^{(2)} \Delta c_{n+1} + S_n^{(1)} c_{n+1}$$

. . . . . .

An obvious induction proof now leads to the fundamental relation

$$(3) \quad S_n = (-1)^k \sum_{i=1}^{n} S_i^{(k)} \Delta^k c_i + (-1)^{k-1} S_n^{(k)} \Delta^{k-1} c_{n+1}$$

$$+ (-1)^{k-2} S_n^{(k-1)} \Delta^{k-2} c_{n+1} + \cdots - S_n^{(2)} \Delta c_{n+1} + S_n^{(1)} c_{n+1}.$$

In order to prove our theorem we must show that S_n approaches a finite limit as n approaches infinity. To do this, it is sufficient to prove that each term in the second member of (3) approaches a finite limit as n approaches infinity.

That the first term approaches a finite limit follows from the convergence of (2) and the fact that (in consequence of the existence and finite value of the first limit in (1)) a positive constant A exists such that $|S_i^{(k)}| < A$ for every i; for then it is clear that the series

$$\sum_{i=1}^{\infty} S_i^{(k)} \Delta^k c_i$$

converges absolutely.

Now the series

$$\sum_{i=1}^{\infty} \Delta^k c_i \equiv \sum_{i=1}^{\infty} (\Delta^{k-1} c_{i+1} - \Delta^{k-1} c_i)$$

converges; and hence

$$\lim_{n=\infty} \Delta^{k-1} c_n$$

exists and is finite. This, in connection with the existence and finite value of the first limit in (1), leads to the conclusion that the second term of the right member of (3) approaches a finite value as n approaches infinity.

Since $\lim_{n=\infty} \Delta^{k-1}c_n$ exists and is finite it follows from Lemma II that each of the limits

$$(4) \qquad \lim_{n=\infty} \frac{\Delta^{k-2}c_n}{n}, \quad \lim_{n=\infty} \frac{\Delta^{k-3}c_n}{n^2}, \quad \cdots, \quad \lim_{n=\infty} \frac{\Delta^0 c_n}{n^{k-1}}$$

(where $\Delta^0 c_n \equiv c_n$) exists and is finite. To see this, let us write

$$u_n^{(t)} = \Delta^{k-t}c_n,$$

where t is one of the numbers $t = 2, 3, \cdots, k$. Then

$$\Delta^{t-1}u_n^{(t)} = \Delta^{k-1}c_n.$$

A direct application of the lemma (for each value of t) now leads to the truth of the statements which we desired to establish.

We retain the same range for the variable t and write

$$(5) \quad S_n^{(k-t+1)}\Delta^{k-t}c_{n+1} = n^{t-1}S_n^{(k-t+1)} \cdot \frac{\Delta^{k-t}c_{n+1}}{(n+1)^{t-1}} \cdot \frac{(n+1)^{t-1}}{n^{t-1}}.$$

If n approaches infinity, it is clear (in consequence of (1) and the results just established) that each of the three factors of the second member of (5) approaches a finite limit. From this we see that every term in the right member of (3) past the second approaches a finite limit as n approaches infinity.

This completes the proof of the theorem.

Another valuable theorem may be obtained by making the restrictions on the c's somewhat stronger and weakening those on the a's. The special case of this theorem when $k = 1$ is also due to Dedekind (l. c., page 371). The general theorem may be stated as follows:

THEOREM II. *Retaining the notation of Theorem I, let us suppose that a positive constant B exists such that each of the quantities*

$$S_n^{(k)}, \quad nS_n^{(k-1)}, \quad n^2S_n^{(k-2)}, \quad \cdots, \quad n^{k-1}S_n^{(1)}$$

is in absolute value less than B for every value of n. Suppose further that the series

$$\sum_{i=1}^{\infty} |\Delta^k c_i|$$

is convergent and

$$\lim_{n=\infty} \Delta^{k-1} c_n = 0.$$

Then the series

$$a_1 c_1 + a_2 c_2 + a_3 c_3 + \cdots$$

is convergent (but not necessarily absolutely convergent).

As before, we have the fundamental relation (3). It is sufficient to show that each term in the second member of (3) again approaches a finite limit. The first term may be dealt with precisely as in the preceding argument. The second term approaches zero, as one sees directly from the hypothesis of the theorem. Since $\lim_{n=\infty} \Delta^{k-1} c_n = 0$ it follows from Lemma II that each of the limits in (4) is zero. Hence, from (5) it follows that every term in the right member of (3) after the second approaches zero as n approaches infinity. This completes the proof of the theorem.

As an example illustrative of the theory, let us consider the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n+1}{n(n+1)} c_n(\rho) \qquad (\rho > 0), \tag{6}$$

where

$$c_n(\rho) = 1 + \frac{1}{2^\rho} + \frac{1}{3^\rho} + \cdots + \frac{1}{n^\rho}.$$

Let the constant c_n of theorem II be $c_n(\rho)$. Then

$$\Delta c_n = \frac{1}{(n+1)^\rho}, \quad \Delta^2 c_n = \frac{1}{(n+2)^\rho} - \frac{1}{(n+1)^\rho} = \frac{-\rho}{n^{\rho+1}} + \cdots;$$

whence it follows that $\Sigma |\Delta^2 c_i|$ converges and $\lim_{n=\infty} \Delta c_n = 0$. Thus c_n satisfies the conditions requisite for applying Theorem II when $k = 2$.

Put

$$a_1 = \frac{1}{2}, \quad a_n = (-1)^{n-1} \frac{2n+1}{n(n+1)} \qquad (n > 1).$$

Then series (6) is identical with $a_1 c_1 + a_2 c_2 + a_3 c_3 + \cdots$ except for the first term; hence it is sufficient to prove the convergence of the last series. We have

$$S_1^{(1)} = \frac{1}{2}, \quad S_n^{(1)} = \frac{1}{2} - \frac{5}{2\cdot 3} + \frac{7}{3\cdot 4} - \cdots + (-1)^{n-1}\frac{2n+1}{n(n+1)}$$

$$= \frac{1}{2} - \left(\frac{1}{2}+\frac{1}{3}\right) + \left(\frac{1}{3}+\frac{1}{4}\right) - \cdots + (-1)^{n-1}\left(\frac{1}{n}+\frac{1}{n+1}\right)$$

$$= (-1)^{n-1}\frac{1}{n+1} \qquad (n > 1);$$

$$S_n^{(2)} = \frac{1}{2} - \frac{1}{3} + \cdots + (-1)^{n-1}\frac{1}{n+1}.$$

It is clear that $S_n^{(1)}$ and $S_n^{(2)}$ satisfy the conditions requisite for the application of Theorem II.

Hence we conclude the convergence of series (6).

§ 3. *Two Theorems on Uniform Convergence.*—By employing the corollaries to Lemmas I and II instead of these lemmas themselves and making certain obvious verbal changes in the proofs as given in § 2, it is easy to establish the following two theorems:

THEOREM III. *Let $a_1, a_2, a_3, \cdots$ be any infinite sequence of numbers such that each of the limits*

$$\lim_{n=\infty} S_n^{(k)}, \quad \lim_{n=\infty} nS_n^{(k-1)}, \quad \lim_{n=\infty} n^2 S_n^{(k-2)}, \quad \cdots, \quad \lim_{n=\infty} n^{k-1} S_n^{(1)}$$

exists and is finite, where $S_n^{(1)}, \cdots, S_n^{(k)}$ have the same meaning as in Theorem I. Let $c_1, c_2, c_3, \cdots$ be an infinite sequence of functions of the complex variable x regular in a given closed domain D and such that

$$\sum_{i=1}^{\infty} |\Delta^k c_i|$$

converges uniformly in D.

Then the series

$$a_1c_1 + a_2c_2 + a_3c_3 + \cdots$$

converges uniformly in D.

THEOREM IV. *Retaining the same notation as in Theorem III, let us suppose that a positive constant B exists such that each of the quantities*

$$S_n^{(k)}, \quad nS_n^{(k-1)}, \quad n^2S_n^{(k-2)}, \quad \cdots, \quad n^{k-1}S_n^{(1)}$$

is in absolute value less than B for every value of n. Suppose further that the series

$$\sum_{i=1}^{\infty} |\Delta^k c_i|$$

converges uniformly in D and

$$\lim_{n=\infty} \Delta^{k-1} c_n = 0.$$

Then the series

$$a_1c_1 + a_2c_2 + a_3c_3 + \cdots$$

converges uniformly in D.

For the case $k = 1$ both of these theorems are already known. This special case of Theorem III appears to have been first employed by Nielsen*; but Nielsen's statement of it is not entirely accurate, as Landau† has pointed out. The corresponding special case of Theorem IV is due to Cahen.‡

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A TRANSLATION PRINCIPLE CONNECTING THE INVARIANT THEORY OF LINE CONGRUENCES WITH THAT OF PLANE n-LINES.

BY PROFESSOR O. E. GLENN.

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A WELL known translation principle of Clebsch§ enables us to construct a ternary invariant by a simple algebraical operation upon a binary invariant. In brief, if the line $u_x = 0$ intersects the curve

$$f = a_x^n = b_x^n = \cdots = 0$$

in the n points given by the binary form

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* *Annali di Matematica* (3), 15 (1908), pp. 275–282.
† *Sitzungsber. d. K. Bayer. Akad. d. Wiss., Phys.-Math. Classe*, 1909.
‡ *Annales scientifiques de l'Ecole Normale Supérieure* (3), 11 (1894), p. 79.
§ Clebsch, Vorlesungen über Geometrie, vol. 1, p. 276; and *Journal für Math.*, vol. 59 (1861).

$$g = a_\xi'^m = b_\xi'^m = \cdots = 0,$$

and if

$$I = \Sigma C(a'b')(a'c') \cdots$$

is any invariant of g, then

$$I(u) = \Sigma C(abu)(acu) \cdots$$

is a contravariant of f; and it represents the envelope of $u_x = 0$ when the latter cuts $f = 0$ in a range having the projective property $I = 0$.

Let us, for the moment, call $f = 0$, $g = 0$ the higher and lower locus, respectively, and $u_x = 0$ the joining locus. Then a translation principle is an algebraical process depending upon the joining locus, which converts an invariant of the lower locus into an invariant of the higher locus.

In this note we establish a translation principle in which the higher locus is a line congruence in three-space, the lower locus a plane n-line, and the joining locus is a plane.

§ 1. *The Congruence* $C(n; 1)$.

The line determined by two planes

$$u_x \equiv x_1 - p_1x_3 - q_1x_4 = 0, \quad v_x \equiv x_2 - p_2x_3 - q_2x_4 = 0$$

was designated by Plücker* as his element of three-space. As coordinates of one of the ∞^4 elements we have the non-homogeneous system

$$(p_1, q_1, p_2, q_2, r) \qquad [r = (pq)].$$

A non-homogeneous form in these variables represents ∞^3 elements forming a line complex. Consider two such forms of orders n and unity respectively,

$$f_n(p, q, r) = 0,$$

$$(1) \qquad f_1(p, q, r) \equiv \sum_{i=1}^{2} (\alpha_i p_i + \beta_i q_i) + \gamma_1 r + \gamma_2 = 0;$$

the ∞^2 elements common to these two complexes form a line congruence which we may represent by $C(n; 1)$.

The intersection of an arbitrary plane $w_x = 0$ *and the congruence* $C(n; 1)$ *is an* n*-line.*

* Plücker, System der Geometrie des Raumes u. s. w. (1846).

An analytical proof is furnished as follows: Impose the condition that $w_x = 0$ pass through the element (u_x, v_x). This is

$$(2) \qquad \begin{Vmatrix} w & w_2 & w_3 & w_4 \\ -1 & 0 & p_1 & q_1 \\ 0 & -1 & p_2 & q_2 \end{Vmatrix} = 0,$$

or

$$(3) \qquad p_1w_1 + p_2w_2 + w_3 = 0, \quad q_1w_1 + q_2w_2 + w_4 = 0.$$

From (3) we get

$$(4) \qquad r \equiv (pq) = \frac{1}{w_1}(w_4p_2 - w_3q_2),$$

and the substitution of this in f_1 eliminates r and reduces f_1 to a linear non-homogeneous form in p_i, q_i $(i = 1, 2)$, involving the parameters w.

We now solve $f_1(p, q)$ with (3), thus expressing p_2, q_1, q_2 each as a linear expression in p_1, say

$$(5) \qquad \begin{aligned} p_2 &= g(w)p_1 + h(w), \\ q_i &= g_i(w)p_1 + h_i(w) \qquad (i = 1, 2). \end{aligned}$$

Substitute these in $f_n(p, q)$ and we get a non-homogeneous equation or form $F(p_1)$ in p_1, with coefficients rational in w_i $(i = 1, 2, 3, 4)$, and of order n. The n roots $p_1^{(j)}$ $(j = 1, 2, \cdots, n)$ put in (5) in turn, give us just n sets

$$(p_i^{(j)}, q_i^{(j)}, r^{(j)}) \quad (i = 1, 2; j = 1, 2, \cdots, n).$$

These sets are the coordinates of n lines of intersection as stated.

Now if we project the n-line in which $w_x = 0$ cuts $C(n; 1)$ upon the $x_3 = 0$ plane, we get the n-line whose equation is

$$(6) \qquad f_{3n} \equiv \prod_{j=1}^{n} (p_2^{(j)}x_1 - p_1^{(j)}x_2 + r^{(j)}x_4) = 0.$$

In the translation principle to be established $C(n; 1)$ figures as the higher locus, $f_{3n} = 0$ as the lower locus, and $w_x = 0$ the joining locus.

§ 2. *Translation Principle.*

It is easy to show by combining (4), (5), (6) that the n lines of f_{3n} are concurrent. Some of the full invariants of

this n-line therefore vanish identically. Thus when $n = 3$ the only full invariant, the condition for a triple point, vanishes identically. Now, however, even when $n = 3$ or $n = 2$ we always have non-vanishing invariants of the binary form $F(p_1)$, and these, through (6), have interpretations as invariants of the lower locus f_{3n}. For instance, if the discriminant of $F(p_1)$ vanishes, then f_{3n} has a double line.

Every rational invariant I of $F(p_1)$ is a rational form in w_i $(i = 1, 2, 3, 4)$. Such a form represents an algebraical surface whose equation is in planar coordinates. *This surface,* $\varphi(w) = 0$, *is the envelope of the plane* $w_x = 0$ *when the latter moves so as to cut the congruence* $C(n; 1)$ *in the n-line having the projective property* $I = 0$.

It is to be noted that q_1, q_2, p_2 each satisfy an equation analogous to $F(p_1) = 0$. But each equation may be obtained from $F(p_1) = 0$ by a homographic transformation of $F(p_1)$, as is readily shown from (1), (3), (4). The determinants of these transformations will be different from zero, save when (1) and (3) are inconsistent. Hence the surfaces obtained from the equations in q_i, p_2 coincide with $\varphi(w) = 0$. In other words the translation principle gives a unique result, $\varphi(w)$.

§ 3. *Class of* $C(n; 1)$, *etc.*

A conspicuous case of I is the discriminant of $F(p_1)$. Applying the translation principle, we get a surface $\varphi(w) = 0$ which is the envelope of $w_x = 0$ when the latter moves so as to touch $C(n; 1)$ in a double line. In this case $\varphi(w) = 0$ may be called the class equation of the congruence $C(n; 1)$. The order of $\varphi(w)$ is the class of $C(n; 1)$.

For illustration we take the special congruence given by

$$(7) \qquad \begin{aligned} f_2(p, q) &\equiv (pq) + p_1q_1 + ap_2 = 0, \\ f_1(p, q) &\equiv (pq) + 1 = 0. \end{aligned}$$

By (3), (4) we have

$$\begin{aligned} w_4p_2 - w_3q_2 + w_1 &= 0, \\ w_1p_1 + w_2p_2 + w_3 &= 0, \\ w_1q_1 + w_2q_2 + w_4 &= 0. \end{aligned}$$

Hence

$$(8) \quad F(p_1) \equiv w_2w_4p_1^2 - (w_2^2 + aw_1w_3)p_1 - w_3(w_2 + aw_3) = 0,$$

and the class equation of the congruence reduces to

$$(9) \qquad \varphi(w) \equiv (w_2^2 + aw_1w_3)^2 + 4w_2w_3w_4(w_2 + aw_3) = 0.$$

It is known that every linear complex for which $(pq) \neq 0$ can be reduced* by projective transformation to the canonical form $(pq) + 1 = 0$. Hence we may take as a general case the congruence given by the general $f_n(p, q)$ and the special f_1 of (7).

It follows that $F(p_1)$ in general has coefficients which are homogeneous expressions of order $2n$ in w_i $(i = 1, 2, 3, 4)$. Hence since the discriminant of $F(p_1)$ is of degree $2(n - 1)$, *the class of the general congruence* $C(n; 1)$ *is* $m = 4n(n - 1)$.

In the special case above the class has been reduced by dividing out powers of w_i.

§ 4. *Generalization to Congruence* $C(n; m)$.

Instead of the non-homogeneous system of Plücker (p_i, q_i, r) we may use the homogeneous coordinates of Grassmann, viz.,

$$(q_{12}, q_{13}, q_{14}, q_{23}, q_{42}, q_{34}),$$

where

$$q_{12}q_{34} + q_{13}q_{42} + q_{14}q_{23} = 0.$$

Then a congruence $C(n; m)$ is given by the two homogeneous equations

$$(10) \qquad f_n(q_{ik}) = 0, \qquad g_m(q_{ik}) = 0.$$

To obtain the lines in which an arbitrary $w_x = 0$ intersects $C(n; m)$ let u_x be an arbitrary plane cutting w_x in an element of the complex f_n. Then

$$q_{ik} = u_iw_k - u_kw_i,$$

and the planar equation of the lines of the complex which are contained in $w_x = 0$ is

$$(11) \qquad f_n(u_iw_k - u_kw_i) = 0,$$

where u_j are the variables. Likewise $w_x = 0$ intersects the other complex in the lines

$$(12) \qquad g_m(u_iw_k - u_kw_i) = 0.$$

The lines (11) envelope a curve of class n† and (12) a curve of

* Lie, Die Berührungstransformationen, p. 223.

† Lie, ibid., p. 286.

class m. Hence $w_x = 0$ *cuts* $C(n; m)$ *in an* nm*-line*, by Bezout's theorem.*

If we project the curves (11), (12) upon the x_3 plane, we may obtain the equation of the projected mn-line by interchanging point and line coordinates in Clebsch's proof of Bezout's theorem (see Vorlesungen über Geometrie, page 282). Every full invariant of this mn-line gives by our translation principle an equation of condition among the coordinates w_i.

An alternative method of procedure is to use equations (1), (3), replacing f_1 by g_m in (1). Rational elimination processes give a form in each variable p_1, q_1, p_2, q_2 with coefficients rational in w_i. Of these forms that in p_2 is the transformed of the one in p_1, say of $F_1(p_1)$, by a homographic transformation, and that in q_2 is likewise the transformed of the one in q_1, viz. $\varphi_1(q_1)$. But φ_1 is not transformable into F_1. As an invariant of the mn-line of intersection we may then select a simultaneous invariant of the binary forms $F_1(p_1)$, $\varphi_1(q_1)$, and by translation this invariant goes into an equation of condition in the variables w_i, representing a contravariant surface.

UNIVERSITY OF PENNSYLVANIA,
PHILADELPHIA, PA.

SOME MATHEMATICAL BOOKLET SERIES.

Matematica dilettevole e curiosa. Di ITALO GHERSI. Con 693 figure originali dell'Autore. Milano, Ulrico Hoepli, 1913. viii+730 pp. Price L. 9.50.

Wo steckt der Fehler? Trugschlüsse und Schülerfehler. Gesammelt von Dr. W. LIETZMANN und V. TRIER. Mathematische Bibliothek, Nr. 10. Leipzig and Berlin, B. G. Teubner, 1913. 57 pp. Price M. 0.80.

ENGLISH and French mathematical literature is entirely lacking in such admirable booklets dealing with elementary topics, as those which have wide circulation in Germany and Italy.† I refer to the Mathematische Bibliothek of the

* Bezout, Theorié générale des Equations algébriques (1779).

† It may be suggested that the volumes on Elimination by Laurent and on Geometrography by Lemoine, of the excellent "Scientia" series (Gauthier-Villars, Paris) are elementary, but these are only two of a dozen volumes by Appell, Gibbs, Hadamard, Poincaré, etc., which certainly may not be classed in this way. And even these two brochures are more

Sammlung Göschen,* the Lietzmann-Witting Mathematische Bibliothek† and the mathematical volumes of the Biblioteca degli studenti,‡ and the Manuali Hoepli.

The Mathematische Bibliothek contains about 35 volumes ($4\frac{1}{4}$ x $6\frac{1}{4}$ inches; uniform price, $22\frac{1}{2}$ cents), each neatly bound in cloth and containing from 130 to 230 pages. A. Sturm, H. Schubert, M. Simon, O. Th. Bürklen, K. Doehlemann and E. Beutel are among the authors and the volumes treat of History of mathematics, Plane geometry, Descriptive geometry (2 volumes), Determinants, Analytical geometry of the plane, Analytical geometry of space (notably fine figures), Projective geometry, Algebraic curves (2 volumes),§ Insurance mathematics, Vector analysis, Geodesy, Surveying, Astronomy, etc.

Of the Mathematische Bibliothek herausgegeben von W. Lietzmann und A. Witting a dozen volumes have already appeared. They are bound in boards, contain 41 to 93 pages ($4\frac{3}{4}$ x $7\frac{1}{4}$ inches) each, and are of the same uniform price as the Göschen Sammlung before 1913. In this series Wieleitner has written on the Idea of number in its logical and historical development; O. Meissner is author of Theory of probabilities with applications; M. Zacharias wrote the Introduction to projective geometry; Zühlke, Geometrical constructions in a limited plane; Beutel, Squaring the circle.

In the Biblioteca degli Studenti are nearly a score of volumes (4 x $6\frac{1}{4}$ inches; limp covers; single numbers of about 85 pages, 10 cents, double numbers of about 170 pages, 20 cents). They include, Manual of plane trigonometry, Manual of spherical trigonometry, Exercises of elementary geometry, Guide to the resolution of problems in algebra, Principles of perspective, Repertorium of mathematics and elementary physics, etc., and treat of very elementary topics.

Some 40 of the 1,200 odd volumes ($4\frac{1}{4}$ x 6 inches) in the Manuali Hoepli series are of mathematical content. Perhaps the two best known works are the volumes (658+950

advanced in character than any of those in three, and than many of those of the fourth series, about to be considered. The same may be remarked concerning the Cambridge Tracts in Mathematics and Mathematical Physics.

* G. J. Göschen'sche Verlagshandlung, Berlin und Leipzig.

† B. G. Teubner, Leipzig und Berlin, 1912–1913.

‡ Raffaello Giusti, editore, Livorno.

§ The second volume was reviewed by Professor White in this BULLETIN, vol. 19, pp. 417–419, May, 1913.

pages) of Pascal's Repertorio di matematiche superiori* since translated into German and enlarged,† and Pascal's Determinanti e applicazioni, 1897, which three years later was elaborated into a volume of the Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften. Then there are 4 volumes on Algebra, 4 on Arithmetic, 2 on Astronomy, 4 on the Calculus, including volumes on Calculus of variations and Finite differences,‡ and Critical exercises on the differential and integral calculus; 1 on Mathematical formulæ;§ Saccheri's Euclide emendato; 3 volumes on Functions (analytic, elliptic, polyhedral and modular‖); 13 on Geometry, 1 on the Mathematics of economics,¶ 1 on Groups, 4 on Mechanics, and 1 by G. Loria on Exact science in ancient Greece.** The volume of Ghersi under review is the second he has written for this mathematical series, the earlier one having dealt with Methods for resolving problems of elementary geometry.

In recent times English, French, and German writers have published popular works for recreation hours of those who are in any wise interested in mathematics. Ball's Mathematical Recreations and Essays, which has recently reached a fifth edition,†† is almost a classic in its special field. The older works of Lucas, "Récréations mathématiques"‡‡ and "L'Arithmétique amusante"§§ are frequently referred to,

* Milano, 1898 and 1900. The first volume is reviewed by E. O. Lovett in this BULLETIN, vol. 5, pp. 357–362, April, 1899.

† Two volumes, Leipzig, 1900, 1901. New edition to be completed in 4 volumes; vols. 1_1, 2_1, 1910, reviewed in this BULLETIN by C. H. Sisam, vol. 19, pp. 372–374, April, 1913.

‡ Translated into German by A. Schepp. Leipzig, 1899. Reviewed by J. K. Whittemore in this BULLETIN, vol. 6, pp. 352–4, May, 1900.

§ This volume by Rossotti was reviewed by E. O. Lovett in this BULLETIN, vol. 5, pp. 261–2, Feb., 1899.

‖ This volume by Vivanti was reviewed by J. I. Hutchinson in this BULLETIN, vol. 14, pp. 144–5, Dec., 1907. The French edition by A. Cahen was reviewed by G. A. Miller in this BULLETIN, vol. 19, pp. 534–5, July, 1913.

¶ This volume by Virgilii was reviewed by J. M. Gaines in this BULLETIN, vol. 5, pp. 488–9, July, 1899.

** This work, which has just been published, 1914, contains about 1000 pages. The title page with "seconda edizione totalmente riveduta" is misleading, as the original work of over 900 pages of *quarto* format was a reprint of memoirs (in five books) in *Mem. Acc. Modena* (2), vol. 10, pp. 3–168; vol. 11, pp. 3–237; vol. 12_2, pp. 3–411; (1893–1902). Futhermore, "Libro II, Il periodo aureo della geometria Greca" appeared in still another form in *Mem. Reale Acc. d. Sc. di Torino* (2), vol. 40, pp. 369–445; (1890).

†† London, 1911.

‡‡ Paris, T. I, 2e éd., 1891; T. II, 2e éd., 1896; T. III, 1893; T. IV, 1894.

§§ Paris, 1895.

while the circulation of Ahren's Mathematische Unterhaltungen und Spiele* and Schubert's Mathematische Musestunden† is confined more to Germany. Each work has its own peculiar ideals, but Ball is perhaps the most comprehensive in range, while he and Ahrens alone introduce, to an appreciable extent, references to the widely scattered literature of the subject. E. Fourrey's "Curiosités géométriques"‡ is also notably full in exact statement of authorities.

From works such as these, from books like Blythe's on Models of cubic surfaces, Catalan's Théorèmes et problèmes de géométrie élémentaire, Cremona's Elementi di geometria proiettiva, Enriques' Questioni riguardanti la geometria elementare, de Longchamps' Essai sur la géométrie de la règle et de l'équerre, Loria's Spezielle algebraische und transzendente ebene Kurven, and from various periodicals, Ghersi has compiled the present little work on Matematica dilettevole e curiosa.

The first 74 pages are taken up with "Curious and bizarre problems" such as: Euler's problem of the Königsberg bridges, the Hampton Court maze and other unicursal problems, map-coloring problem, and chess problems. Of course little more than the statement of a problem is frequently given.

In the next 100 pages various curious properties of numbers, and problems of arithmetic and arithmetic geometry are set forth. For example, we have properties of perfect and amicable numbers, of the triangle of Pascal, of Lucas's singular products, as well as problems of Benedetti (Speculationes diversæ, 1585) and of Leonardo Pisano (Liber Abaci, 1202).

Fermat's equation and other problems of the theory of numbers are treated in the next 15 pages, then follows a collection of miscellaneous algebraic problems which conclude with graphical solutions of equations of the second, third and fourth degrees and with a sketch of Demanet's and Meslin's hydraulic,‡ and Lucas's electric solution of equations.

Magic squares, magic polygons, and magic polyhedra are illustrated on pages 251–326.

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* Leipzig, 1901.

† Grosse Ausgabe, Leipzig, Bd. I, 3. Aufl., 1907; Bd. II, 2. Aufl., 1900.

‡ 2e éd., Paris, 1906.

§ Cf. "Two hydraulic methods to extract the nth root of any number" and "Hydraulic solution of an algebraic equation of the nth degree," by A. Emch, *American Mathematical Monthly*, January and March, 1901, vol. 8, pp. 10–12 and 58–9.

Then follow 350 pages treating of miscellaneous questions in geometry. On pages 329–367 we find definitions and derivation of properties, of notable transcendental, and cubic, quartic, and other algebraic curves. The next dozen pages contain instruments for tracing by continuous motion such curves as the conic sections, cissoid, and conchoid. Some 20 pages given over to discussion of the solution of problems in elementary geometry, by ruler and compass, and then (pages 407–422) cyclotomy is touched upon. Then come 100 pages occupied with the problems of trisection of an angle, squaring the circle, duplication of the cube. Dissection of figures, geometrical pavements, star-polyhedra, and hyperspace are some of the concluding topics under the head geometry.

In the final sections are paradoxes and other recreations in mechanics.

It will be remarked, as indeed the title implies, that the volume is not confined to so-called recreations, although these occupy the major part of the volume. It is written with light touch and anyone unacquainted with books on mathematical recreations may pass a few pleasant hours in turning over the pages and find some things not met with in other books of the kind. The reader who wishes to learn more of the underlying theory will then naturally turn for guidance to such a book as Ball's or to the article in the Encyklopädie* or to such works in fields other than those of recreations, as mentioned above.

In the Lietzmann-Trier Bändchen, which may be classed as a small addition to the literature of mathematical recreations, Lietzmann collected the 36 fallacies (Trugschlüsse) and Trier the 50 pupils' mistakes. Arithmetic, algebra, elementary geometry (synthetic and analytic), trigonometry are the only subjects illustrated. The errors in the reasoning are not indicated.

Among the fallacies are (1) numerous examples depending for their results upon division of each side of an equality by zero or neglect of consideration of double sign before a radical; (2) a series of geometrical paradoxes, several of which are already familiar through Ball's book.

* Mathematische Spiele, von W. Ahrens, vol. I, 2, pp. 1080–1093, Leipzig, 1902.

Here is an example of a different kind, which appears to be new: "Consider

$$(1) \qquad \log_e 2 = 1 - 1/2 + 1/3 - 1/4 + 1/5 - \cdots;$$

multiplying through by 2 we get

$$2 \log_e 2 = 2 - 1 + 2/3 - 1/2 + 2/5 - 1/3 + 2/7 - \cdots.$$

Collecting terms with common denominators and arranging according to increasing denominators, we get

$$(2) \qquad 2 \log_e 2 = 1 - 1/2 + 1/3 - 1/4 + 1/5 - \cdots.$$

This is, however, the same as (1). Therefore

$$\log_e 2 = 2 \log_e 2."$$

The examples of Schülerfehler are taken from the exercises of Danish pupils. The vagaries of American youth suggest that an equally interesting collection could be made on this side of the water. The error in No. 32 is not evident. But here is No. 36: "Given two circles which cut one another in P and Q and touch the sides of an angle, on the same side of the vertex, at the points A, A_1 for one circle and B, B_1 for the other. Prove (1) that PQ produced passes through the middle points of AB and A_1B_1; (2) that AA_1, BB_1 and PQ are parallel to one another." Solution: "PQ cuts AB in C, A_1B_1 in C_1. Then by the power theorem, $CA^2 = CP \cdot CQ = CB^2$. Therefore C is the middle point of AB. In the same way C_1 is the middle point of A_1B_1. AA_1, PQ and BB_1 are parallel to one another because they cut off the equal segments on the lines AB and A_1B_1."

Finally, No. 47: "The sides of a triangle are a, b, c. To express sin A in terms of the given quantities." Solution: "Of course the following relations hold good:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

In a proportion it is allowable to interchange the means; hence

$$\frac{a}{\sin A} = \frac{b}{c} = \frac{\sin B}{\sin C}. \qquad \therefore \sin A = \frac{ac}{b}."$$

R. C. ARCHIBALD.

BROWN UNIVERSITY,
PROVIDENCE, R. I.

MATHEMATICAL MODELS.

Catalog mathematischer Modelle für den höheren mathematischen Unterricht. Veröffentlicht durch die Verlagshandlung von Martin Schilling in Leipzig, mit 106 Abbildungen. Siebente Auflage, Leipzig, 1911, xiv + 172 pp.

Verzeichnis von H. Wieners und P. Treutleins Sammlung mathematischer Modelle für Hochschulen, höhere Lehranstalten und technische Fachschulen. Zweite Ausgabe mit 6 Tafeln. Leipzig und Berlin, Verlag von B. G. Teubner, 1912, 64 pp.

Abhandlungen zur Sammlung mathematischer Modelle. In zwanglosen Heften herausgegeben von HERMANN WIENER. Leipzig, Verlag von B. G. Teubner. 1. Heft von H. Wiener, 1907, 90 pp.; 2. Heft von P. Treutlein, 1911, 20 pp.

Illustrierter Spezialkatalog mathematischer Modelle und Apparate. Entworfen von G. KOEPP und anderen bewährten Fachmännern. New York City, Eimer and Amend, 128 pp.

FOR fifteen years the bulk of the models used by advanced mathematical students all over the country has been procured from Schilling of Leipzig. This firm was developed from the Firma L. Brill of Darmstadt, the foundation of which reaches back some 35 years. Klein and A. von Brill, in those early years professors in the Technische Hochschule at Munich, had more than two score of models made for the Hochschule under their direction. Copies of these (for example: the tractrix of revolution, geodetic lines on an ellipsoid of rotation, Kummer's surface, forms of Dupin's cyclide, the spherical catenary, twisted cubics) in gypsum, wire, and brass form a portion of the great Schilling collection. In more recent times construction of new models has been carried on by the assistance of many other mathematicians. Among them are Professors Dyck, Finsterwalder, Kummer, Schoenflies, H. A. Schwarz, C. und H. Wiener.

As nearly ten years had passed since the sixth edition of the catalogue, the seventh edition† fills a long felt want. It

† Descriptions of models which have been manufactured since this edition was published, have appeared in *Jahresbericht d. Deutsch. Math.-Ver.*, 1913, vol. 22, pp. *75–76, 134–137.*

describes some 400 models. Nothing more than an indication of the subjects illustrated can be given here: surfaces of the second order, algebraic surfaces of the third order, algebraic surfaces of the fourth and higher orders, line geometry, screw surfaces, space curves and developable surfaces, descriptive and projective geometry, analysis situs, algebra, function theory, mechanics and kinematics, mathematical physics, and structure of crystals,

Shortly after the third International Mathematical Congress at Heidelberg in 1904, Teubner offered to the public a selection of about 60 of the mathematical models for Hochschule instruction which had been exhibited at the Congress by the mathematical Institut of the Technische Hochschule of Darmstadt. The construction of the models in the selection was inspired by Professor H. Wiener.* In the new catalogue now before us we find that Professor Wiener has increased his collection by 50 models, while the late Professor Treutlein has contributed about 200 more.† All of the models are designed as aids to instruction in German secondary schools and Hochschulen. For students of higher mathematics the models of twisted curves and deformable quadric surfaces will probably be the only ones of especial interest.

The Abhandlungen are intended to be of value for those using the models. In Heft 1 are 9 Abhandlungen by Wiener: (1) Mathematical models and their use in instruction (pages 3–8); (2) On the projection of some plane figures (9–10); (3) The regular Platonic polyhedra, Regularity in a group (11–14); (4) Regular polygons and closed reflective systems (15–18); (5) The building up of the regular polyhedra (19–51); (6) How shall surfaces, especially those of the second order, be drawn? (52–54); (7) On surfaces of the second order (55–84); (8) Deformable thread models of ruled surfaces of the second order with fixed thread lengths (85–87); (9) Deformable metal-bar models for transforming a surface of the second order into confocal surfaces (88–91).

These Abhandlungen are similar to those which Schilling

* The catalogue (Verzeichnis mathematischer Modelle, 28 pp.) was published in 1905.

† An interesting account of these models written by Prof. H. Wiener, may be seen in *Jahresbericht d. Deutsch. Math.-Ver.*, Nov., 1913, vol. 22 pp. 297–304.

distributes* with his models and some of them are of considerable interest; on the one hand because of the developments of the theory of the surfaces, on the other through the application of the theory to construction of the models. Numerous bibliographical references are given. In illustration of these characteristics note, for example, (5) and (9).

In (5) the first five pages contain an historical review of the subject, then the theoretical considerations are treated under the headings: The notion of a polyhedron (e. g., Idea of a side, of "Vielkant," of "Vielflach," of "Vielzell"); First and second definitions of the regular polyhedron by the group; Transformation of an angle into a neighboring angle (by rotation); Range of the different suppositions; Third definition of the regular polyhedra; Construction of a regular polyhedron from its group.

In (9) we find that the construction of the model was made possible through theorems of Henrici and Greenhill. Among other studies in this connection, those of Mannheim, Darboux, and Schur are also considered.

The second Heft, written by Treutlein, contains Abhandlungen on the following subjects: "On the intuitive method of mathematical instruction" (pages 3–6); "On mathematical models and their use in teaching" (7–9); "Explanations in connection with the series, and the single models, of the Treutlein collection" (10–20).

In all of the above mentioned publications, Dyck's Katalog† is frequently referred to.

The Eimer and Amend collection is of use more particularly in connection with elementary work in planimetry, stereom-

*The "Erste Folge, Abhandlungen zu den Serien I-XXIII, mit 71 Figuren auf 6 Tafeln und im Text" have also been published in a single volume, in connected form. In the "Neue Folge" Hefte 1-9 have been already issued between 1899 and 1912. The authors are: Fr. Schilling, H. Wiener, W. Ludwig, H. Grassmann, W. Boy, E. Estanave, R. Hartenstein, and F. Pfeiffer.

†Katalog mathematischer und mathematisch-physikalischer Modelle, Apparate und Instrumente. Unter Mitwirkung zahlreicher Fachgenossen herausgegeben im Auftrage des Vorstandes der Deutschen Mathematiker-Vereinigung von geh. Hofrat Dr. Walther v. Dyck, Professor an der Technischen Hochschule zu München. Teubner, Leipzig, 1892, xvi+430 pp.

Nachtrag, Leipzig, 1893, x+135 pp.

These volumes contain papers by Klein, Voss, Brill, Hauck, v. Braunmühl, Boltzmann, Amsler, and Henrici, beside descriptions of the various models by their respective designers.

etry, trigonometry, and related branches. The models of star-polyhedra, Poinsot polyhedra, and the so-called Archimedian semi-regular solids may be mentioned as desirable for more advanced mathematical considerations.

R. C. ARCHIBALD.

BROWN UNIVERSITY,
PROVIDENCE, R. I.

DIFFERENTIAL GEOMETRY.

Leçons sur les Systèmes orthogonaux et les Coordonnées curvilignes. Par GASTON DARBOUX. Deuxième édition, complétée. Paris, Gauthier-Villars, 1910. 8vo. i+567 pp.

THE first 323 pages of the present volume constitute a reprint of the first edition, which was reviewed in the BULLETIN of January, 1899, by President E. O. Lovett. It was originally Darboux's intention to include in the complete work a number of other subjects, such as the theory of quadratic differential forms. This plan was eventually abandoned, and only such subjects are discussed in the completed volume which is now before us as are more or less directly connected with orthogonal triple systems of surfaces and families of Lamé. From the point of view of the artistic unity of the book, this change of plan is only to be commended. Let us hope however that the Fates may deal kindly with the great master of differential geometry; may the fear, which he expresses in his preface, of being prevented from completing his discussion of these other matters, prove to be unfounded; and may he be permitted to add many further contributions to the science to which he has devoted his life and which already owes him so much.

For mathematics owes a great debt to Darboux. Rarely do we find such a combination of the geometer and the analyst as is present in this master mind. He makes clear, not merely by precept but by example, that it is not enough to express a problem of geometry in analytic form and then solve the differential equations. He recognizes the true task of differential geometry as an exhibition of the complete parallelism between analysis and geometry. Not until this has been accomplished does he rest content with the solution of a problem, and many of his most notable contributions have resulted from this tendency to strive for a complete

understanding of a mathematical situation from more than one point of view.

The present volume is divided into three books. The first two of these constitute the subject matter of the first edition, and we may refer for an account of their content to the review by Lovett which has already been mentioned. The third book begins with the exposition of certain existence theorems for systems of partial differential equations, based on the method of successive approximations, which has nowadays become thoroughly familiar to most mathematicians. The systems of partial differential equations considered are all of the form

$$\frac{\partial u_i}{\partial x_h} = f_{ih}(u_k, x_{k'}), \tag{1}$$

where the indices i and h may assume different values and where the right members are given functions of the unknown functions and the independent variables.

Darboux considers three cases. First, let the system be such that one and only one of the partial derivatives of each unknown function is given by an equation of form (1). Then, under certain simple continuity conditions, there exists a unique system of solutions such that u_i reduces to an arbitrary function of all of the x's except x_h, when x_h assumes its initial value.

Darboux's second case is that of a completely integrable system, when *all* of the first order partial derivatives of *each* unknown function are expressible in the form (1), and when the resulting integrability conditions are satisfied.

The third case, intermediate in character, is that of a system in which several, but not all, of the first order partial derivatives of each of the unknown functions are expressible in form (1).

In Chapter II, these theorems are applied to an equation of the form

$$\frac{\partial^2 z}{\partial x \partial y} = f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right).$$

The result (already familiar from a direct investigation by Picard) is this: there exists a unique solution z which reduces to a given function of x for $y = y_0$ and to a given function of y for $x = x_0$. A second application of the results of Chapter I is concerned with a corresponding theorem for the system

$$\frac{\partial^2\theta}{\partial y\partial z} = f_1\left(x, y, z, \frac{\partial\theta}{\partial y}, \frac{\partial\theta}{\partial z}\right),$$

$$\frac{\partial^2\theta}{\partial z\partial x} = f_2\left(x, y, z, \frac{\partial\theta}{\partial z}, \frac{\partial\theta}{\partial x}\right),$$

$$\frac{\partial^2\theta}{\partial x\partial y} = f_3\left(x, y, z, \frac{\partial\theta}{\partial x}, \frac{\partial\theta}{\partial y}\right).$$

Darboux now makes use of these theorems for the solution of some interesting problems of geometry. Let S and S_1 be any two surfaces, and let us establish a point-to-point correspondence between them, such that the tangent planes at any pair of corresponding points M and M_1 are parallel. There will exist in general two tangents Mt and Mu of S, such that the tangents M_1t_1, M_1u_1 of S_1 which correspond to them are respectively parallel to Mt and Mu. Mt and Mu will then be conjugate tangents of S at M, while M_1t_1 and M_1u_1 are conjugate tangents of S_1 at M_1. The curves of S tangent to Mt and Mu, and the curves of S_1 tangent to M_1t_1 and M_1u_1, thus form two conjugate nets which are said to be parallel. Having developed these notions, Darboux formulates the following problem. Given a surface S and a conjugate net traced upon S; to find all of the surfaces S_1 whose points correspond to S in such a way that the corresponding tangent planes are parallel and that to the given conjugate net of S there corresponds on S_1 a conjugate net.

The solution is very simple. Let C and D be the two curves of the original net which pass through M. Let M_1 be any point of space and let C_1 and D_1 be any two curves through M_1 whose tangents are respectively parallel to those of C and D. There exists one and only one surface S_1 passing through C_1 and D_1 and satisfying the conditions of the problem.

The corresponding problem in three dimensions next occupies the attention of the author. If two triple systems of surfaces or, what amounts to the same thing, two systems of curvilinear coordinates of space, can be made to correspond in such a way that at all pairs of corresponding points the tangent planes of the coordinate surfaces are parallel, Darboux speaks of them as parallel systems. It is easy to see that the coordinate curves of each system will then form networks of conjugate curves on each of the coordinate surfaces. Such a triple system of surfaces is said to be a conjugate

system and clearly, on account of Dupin's theorem, every orthogonal system is included in this class.

The existence theorems proved in Chapter I now easily lead to the following result. Let M be a point of space, and let C_1, C_2, C_3 be the three coordinate curves which, in a given conjugate system, pass through M. Choose any other point M' of space and let C_1', C_2', C_3' be any three curves through M' subject only to the condition of having their tangents respectively parallel to those of C_1, C_2, C_3. There exists one and only one conjugate triple system of surfaces parallel to the given conjugate system and having the curves C_1', C_2', C_3' among its coordinate curves.

After these preparations the general conditions for the existence of a conjugate triple system of surfaces become easily intelligible. Let Σ, Σ_1, Σ_2 be three surfaces passing through the point M, and let C_i be the curve of intersection of Σ_k and Σ_l. The three surfaces are subject only to the conditions that the tangents at M to C_k and C_l shall be conjugate tangents of Σ_i. Then it is possible, in an infinite number of ways, to construct on each of the surfaces Σ_i a conjugate net containing the curves C_k and C_l. The choice of these three conjugate nets involves the introduction of three arbitrary functions, each of two independent variables. These having been chosen, there exists one and only one conjugate triple system containing the given three surfaces Σ_i as coordinate surfaces, and the curves of the three arbitrarily chosen conjugate nets on these surfaces as coordinate lines.

The theory of conjugate triple systems is skilfully reduced by Darboux to the consideration of the following systems of partial differential equations:

$$(A) \qquad \frac{\partial \beta_{ik}}{\partial \rho_l} = \beta_{il}\beta_{lk} \qquad (i \neq k \neq l),$$

$$(B) \qquad \frac{\partial U_i}{\partial \rho_k} = \beta_{ik}U_k,$$

$$(B') \qquad \frac{\partial H_i}{\partial \rho_k} = \beta_{ki}H_k, \qquad (i, k, l = 1, 2, 3),$$

$$(C) \qquad \frac{\partial u}{\partial \rho_i} = H_iU_i,$$

where the notation may be explained as follows. Let

$$x = f(\rho_1, \rho_2, \rho_3), \quad y = g(\rho_1, \rho_2, \rho_3), \quad z = h(\rho_1, \rho_2, \rho_3)$$

be the expressions for the cartesian coordinates of a point in terms of the conjugate triple system of curvilinear coordinates under consideration, so that the triple system is composed of the surfaces $\rho_1 = \text{const.}$, $\rho_2 = \text{const.}$, $\rho_3 = \text{const.}$ Let the direction cosines of the coordinate curves be X_i, Y_i, Z_i. Suppose now that a system of six functions β_{ik} of ρ_1, ρ_2, ρ_3 has been obtained which satisfies (A). We must then find three systems of solutions of the three equations (B), which may then be identified geometrically with X_1, X_2, X_3; Y_1, Y_2, Y_3; Z_1, Z_2, Z_3. It remains to find H_1, H_2, H_3 by integrating (B'), and finally to obtain x, y, z by quadratures from (C), identifying in order u with x, y, and z.

It is clear that the integration of system (A) is the most serious part of this general program. Darboux shows, in Chapter III, that there exists a single partial differential equation of the sixth order for a single function V whose integration completely replaces that of system (A) as far as this geometric problem is concerned.

A new notion is now introduced, that of *supplementary* triple systems of surfaces or, as we may say, polar systems. Whenever a conjugate triple system S is given, one may find an infinite number of triple systems Σ, such that the fundamental trihedrals of the two systems at corresponding points are polar to each other, that is, such that the tangents of the coordinates lines of Σ are perpendicular to the tangent planes of the coordinate surfaces of S.

In Chapter IV the author returns to orthogonal triple surfaces. Since these are also conjugate systems they must satisfy the equations (A), (B), (B'), (C). But they must also satisfy the further equations

$$(A') \qquad \frac{\partial \beta_{ik}}{\partial \rho_i} + \frac{\partial \beta_{ki}}{\partial \rho_k} + \beta_{li}\beta_{lk} = 0,$$

and

$$(D) \qquad \frac{\partial U_i}{\partial \rho_i} = -\beta_{ki}U_k - \beta_{li}U_l,$$

which essentially express the fact that, for an orthogonal triple system, the parallel systems and the polar systems are identical. The general existence theorems applied to the

enlarged system of partial differential equations lead to the following result.

Let there be given three surfaces (not planes or spheres), cutting each other in pairs orthogonally along a common line of curvature. There exists one and only one orthogonal triple system containing the given three surfaces. If one of the three surfaces is a plane or sphere, to obtain a correspondingly unique result, there must be given upon it an orthogonal network of curves to take the place of the lines of curvature which in these cases are indeterminate.

Chapter V is given up to a discussion of the theorems of Combescure and Ribaucour which lead to a transformation theory enabling one to deduce from a given triple system an infinite number of others, containing additional arbitrary functions.

The author now takes up the problem anew, from the general kinematic point of view which is characteristic of his theory of surfaces. In this connection he obtains the third order differential equation of Bonnet (who was the first to notice the possibility of such a reduction), and introduces a new set of variables in terms of which the fundamental differential equation assumes a very simple form. Although these new variables are complicated by imaginary quantities, they seem to be especially well adapted for further investigations.

The main body of the book closes with a detailed consideration of triple systems which admit a continuous group of transformations of Combescure.

The four notes which follow are devoted to the application of Abel's theorem to the discovery of *algebraic* triple systems; to the theory of the cyclides of Dupin; to orthogonal triple systems of cyclides; and to the discussion of a special class of point transformations closely connected with the subject of orthogonal triple systems.

We may say that the new portion of this great work is characterized by a generalization of great moment. Orthogonal triple systems now appear as special cases of the more general notion of conjugate triple systems. But the theory of conjugate triple systems belongs properly to projective differential geometry, more specifically to that chapter of projective differential geometry which deals with triple systems of surfaces, a chapter which has been attacked but very recently, in a remarkable Columbia dissertation, by

G. M. Green.* There can be little doubt but that the methods of projective differential geometry will serve to throw a flood of light upon the theory of conjugate triple systems, which promises to become a most fascinating and fruitful field for further research.

E. J. WILCZYNSKI.

THE UNIVERSITY OF CHICAGO,
September 23, 1913.

DESCRIPTIVE GEOMETRY.

Elements of Descriptive Geometry, with applications to spherical and isometric projections, shades and shadows, and perspective. By A. E. CHURCH, late professor of mathematics in the United States military academy, and G. M. BARTLETT, instructor in descriptive geometry and mechanism in the University of Michigan. New York, American Book Company, 1911. 286 pages and 143 figures.

Practical Geometry and Graphics. By D. A. LOW, professor of engineering, East London College. London, Longmans, Green and Company, 1912. 448 pages and 823 figures.

Vorlesungen über darstellende Geometrie. By GUIDO HAUCK, late professor of descriptive geometry and graphical statics in the technical school at Berlin; edited by ALFRED HAUCK, director of the Realschule in Schönlanke. Leipzig, Teubner, 1912. Volume I. 639 pages and 641 figures.

Lehrbuch der darstellenden Geometrie für technische Hochschulen. By EMIL MÜLLER, professor of mathematics at the technical school of Vienna. Second volume, first instalment. Leipzig, Teubner, 1912. 129 pages and 140 figures.

MR. Bartlett's revision of Church's well-known Descriptive Geometry follows the original text very closely, the main departures being in mechanical make-up. The type is larger; the articles are renumbered; important theorems and new words are printed in bold-faced type; the figures are re-drawn and put in the text itself, a number of new ones being added, in particular, two hyperboloids tangent along a generator,

*G. M. Green, "Projective differential geometry of triple systems of surfaces." Dissertation Columbia University, 1913. Press of the New Era Printing Company, Lancaster, Pa.

development of a tea-pot spout, intersection of a torus and a cylinder as exemplified in a joint of a water pipe; a generous list of illustrative exercises for the student is distributed throughout the text. A few corrections in loose mathematical expressions have been made, but too many still remain, and a few of the new ones are also inaccurate. In all important features the text is still elementary and conservative. While it retains many of the antiquated features long discarded in the draughting room, still it does impart the essence of the subject in a lucid and concise way.

Professor Low's book commences with elementary constructions usually met with in books on plane geometry. The chapter on the circle is in sufficient detail for the actual beginner in geometry. Then follows one on geometrical conics, after which comes a short chapter on tracing-paper problems, In the introduction of each new idea the student is urged to perform all the steps of every construction. Chapter V considers approximate solutions of various graphical problems, their actual solution being impossible with ruler and compasses; they include rectification of a circular arc, squaring of the circle, obtaining the square root of π, and others. Roulettes and glissettes are treated by means of a combination of the methods of tracing paper and of approximate solutions. The short chapter on vertors contains little more than what one learns in a first course on mechanics; the principles are applied to graphical statics, including an extensive treatment of center of gravity and of moment of inertia.

Now follows a chapter of 20 pages on plane coordinate geometry, starting at the beginning and treating straight lines, conics, and spirals. A chapter on harmonic motion combines graphical with analytic methods in the discussion of finite series of sine curves.

Many topics are treated more thoroughly than even in the better American institutions; apart from the rule of thumb method of approximations, these 165 pages cover much the same kind of work as an American student would have had who had taken high school mathematics, a brief course in graphical algebra, and an elementary course in mechanics.

The subject proper begins with projection, first illustrated by a drawing in central perspective, then followed by the usual two-plane method of orthographic representation.

Treatment of points, lines, planes, prisms, pyramids, polyhedra, cones, and spheres is rapid, but clear and extensive. In the discussion of the uses of the general profile the drawings in h and v are accompanied in each case by a pictorial one which makes the problem much more real. The elementary principles are followed by numerous practical applications, including joinery, mitred mouldings, and a generous list of problems for the student.

A detailed discussion of the sphere and the quadric cylinder and cone follows, after which we return to the consideration of plane figures defined in various particular ways.

This second part of 114 pages completes the treatment of orthographic two-plane projections. It includes all the principles usually given in an ordinary course in descriptive geometry, treats a number of special cases, provides one or more figures for every case considered, and furnishes hundreds of exercises for the student to work out.

Now follows an excellently well written chapter on horizontal projection, including contours and various problems connected with them. In this chapter a knowledge of differentiation is presupposed. Now comes pictorial drawing, parallel perspective and central perspective, together with the relation between these methods and those of orthographic projection.

A combination of various preceding methods is now employed in the discussion of curved surfaces, tangent planes, developments, helices and screws, intersections of surfaces, together with a short introduction to shades and shadows. The book closes with a short chapter on miscellaneous applications to solid geometry. It is provided with an index and a number of numerical tables.

One who has mastered Professor Low's elaborate treatise will be well equipped with a knowledge of graphical processes and particularly with the relations between them, although no use is made of general axonometric methods; isometric projection is mentioned incidentally as a particular kind of pictorial drawing.

During a long period the lectures and exercises of Professor Guido Hauck at the technical school of Berlin have enjoyed a reputation for elegance, beauty, and pedagogical merit hardly to be found elsewhere. Under these circumstances it

is interesting to be able to see the book prepared on the results of this experience and to profit by the study of the methods of the master. The science of descriptive geometry is essentially an auxiliary one; its main purpose is to provide suitable tools for other purposes, namely, the proper representation of objects for the artisan and the engineer. In so far as it remains a science and does not become purely an empirical procedure it serves the further purpose of providing real mental discipline and spurs the students to originality in extending its scope and its usefulness.

The book commences with the usual presentation of the orthogonal two-plane projection, with the added feature of showing the reader how to distinguish which of two segments lies in front of the other—first of points, then of segments of lines, then of polygons, the latter incidentally determining a plane. True lengths of segments, size of triangles, shape and size of polygons are considered before the plane is mentioned. Then follow a number of problems concerning planes, defined by three non-collinear points, problems involving the angles which a line makes with the vertical and the horizontal planes, distance between parallel planes, etc. A large number of exercises are provided, but all of them are solved in the text, though in many cases not much more than an outline of the solution is given. In the constructions the ground line is always used, but the figures have but very few letters—most of them not containing any, and scarcely any construction lines are provided. This arrangement makes it much harder to understand the solution, but prevents mechanical reproduction. It puts the difficulty in another place by putting the student squarely on his own feet to provide the details himself. In this way the power of space visualization is greatly enhanced, incidentally giving the student the esthetic satisfaction of doing much of the work himself.

The idea of rotating a polygon about one of its sides and of reflection, even of a stereometric figure, about a line are introduced early and used frequently. In the first two chapters these ideas are treated as distinct problems, without much emphasis of the relations between them; the first time any new idea is introduced it is developed in considerable detail, but when use is made of the same idea later the reader is supposed to be familiar with the principle; one who reads without mastering the details will soon be swamped.

Chapter III, stereometric constructions, begins with a resumé of all the results obtained in the preceding ones, then applies them to a number of constructions of solid geometry. The next chapter, called transformations, while dealing with no ideas that are essentially different from those treated by other authors, gives a very systematic development of the use of rotation about certain lines. The first mention of a profile plane is made on page 62 as an incident in the discussion of transformations. Intersections of prisms, pyramids, etc., roof constructions, excavations complete the ordinary two plane discussion of figures bounded by planes.

Now follows a short chapter on axonometric drawing, together with a comparison with the preceding method, showing its advantages and disadvantages; then comes a chapter on certain skew parallel projections, in particular the military and the cavalier. All three methods are illustrated by a complicated architectural drawing.

The graphical development is here interrupted to give a considerable discussion of the elements of projective geometry, showing various particular kinds of perspective, including affinity and similarity and plane perspective. Cross-ratio and harmonic forms are treated metrically and algebraically. The idea of a curve first appears on page 133—the concepts of tangent, contact, curvature, evolute, and involute are all discussed and the statement explicitly made that our only general method of constructing curves is to locate a number of points and then connect them freehand.

No knowledge of conics is presupposed, but an extensive theory is built up, commencing with Steiner's method of generating a conic as locus of intersection of corresponding rays of two coplanar projective pencils, then establishing the theorems of Pascal, Brianchon, and Desargues, the theory of pole and polar, center and diameters, asymptotes, and focal properties. These ideas are then skilfully applied to a large number of problems of construction.

Curves in space are given a much briefer treatment, details being given only for helices. Then follow developables, cones, cylinders, and curves of intersection, with development, surfaces of revolution, quadric surfaces by means of parallel sections, and finally topographical surfaces, conoids and ruled quadrics. In the forthcoming second volume curves of intersection of curved surfaces and shades and shadows are to be treated.

The first volume of Professor Müller's book was reviewed in the BULLETIN, volume 16, page 136. This first instalment of the second volume maintains the same standard and vigor as the first volume, and furnishes a comprehensive compendium as well as a text-book on the subject.

The volume begins with horizontal projection, and applies it to topography, roof construction, and excavations, the scope and treatment being rather similar to that given by Professor Low. Then follows a chapter on axonometry, with application to the representation of curve surfaces, including a number of metrical problems. An interesting feature is the extensive historical development, given in the form of footnotes.

In this set of four books, one American, and three European, we find a good representation of the relative states of the science, as viewed by the different countries. When shall we be able to regard descriptive geometry as a science coextensive with projective and analytic geometry?

VIRGIL SNYDER.

CORNELL UNIVERSITY,
August, 1913.

SHORTER NOTICES.

Zahlentheorie. By KURT HENSEL. Berlin and Leipzig, G. J. Göschen, 1913. 356+xii pp.

As a knowledge of the elements of the theory of congruence of integers is essential in many branches of mathematics, and as the higher parts of the theory of numbers have enthusiastic devotees, it is not surprising that there are published yearly several books treating the theory of numbers from various standpoints. The usual topics on congruences, including the reciprocity law for quadratic residues, are treated in the present book, but at widely separated intervals, the interspersed material being of quite a different nature described below. Consequently, a reader desirous of acquiring rather quickly a knowledge of the classical theory of congruences will not find the present book so well adapted to his needs as most of the texts available. However, there will be readers who appreciate the opportunity of being able to pick up incidentally this useful information while enjoying

Hensel's new and powerful method for number-theoretic questions, a method analogous to that of power series in the theory of analytic functions. This new method in number theory is that of p-adic numbers described in detail in my extensive review, BULLETIN, volume 17 (1910), pages 23–36, of Hensel's Theorie der algebraischen Zahlen. The newer book treats also of the similar, but more general, g-adic numbers, where g is any integer, whereas p was a prime. The g-adic numbers are, however, compounded in a very simple manner of p_i-adic numbers, where $p_1, p_2, \cdots$ are the prime factors of g, so that the matter finally rests upon the case treated in the former book. The new book is however an essentially new contribution to this subject. It gives also an exposition of series of p-adic numbers and of g-adic numbers; as well as of power series in which the variable and the coefficients are p-adic or g-adic numbers, with a detailed treatment of the exponential and logarithmic functions of a g-adic argument. The final chapter (sixty pages) obtains from the point of view of p-adic numbers the more fundamental properties of binary and ternary quadratic forms.

It is certainly instructive to see the theory of congruences developed by the side of, and largely by means of, the theory of p-adic numbers, a theory shown in Hensel's former book to be a powerful instrument for the investigation of algebraic numbers.

L. E. DICKSON.

Leçons sur les Equations intégrales et les Equations intégro-différentielles. Professées à la Faculté des Sciences de Rome en 1910, par VITO VOLTERRA, et publiées par M. TOMASSETTI et F. S. ZARLATTI. Paris, Gauthier-Villars, 1913. vi+164 pp.

THE subject of integral equations is rapidly coming to the front as one of the most important branches of mathematics. Several books on this subject have been published recently, but the book in question, being the published lectures of one of the founders of the theory of integral equations, will no doubt be received with an especially hearty welcome by mathematicians. As stated in the preface, the book is based on the course of lectures given by Volterra at the University of Rome in 1909–1910. The plan of the lectures, however, has been somewhat changed, partly by adding some interesting investiga-

tions of more recent date and partly by leaving out most of the applications. This latter fact seems regrettable, but we are promised by the author a second volume which will be devoted to various applications and also to a more extended presentation of the theory of integro-differential equations.

The object of the author is to give a brief and systematic outline of the entire field of integral equations. The treatment of the topics considered is excellent, and the book will appeal not only to the student who wishes to lay a good foundation for further work but also to the more superficial reader whose object is simply to get some idea of the subject.

The author often confines himself to special cases imposing certain restrictions on the functions with which he is dealing. It might have been desirable to give a more general treatment. In nearly all cases, however, he points out how his methods may be extended to more general cases. Whether this is a good procedure or not is, of course, a matter of opinion. The book covers an unusually large number of topics, and a great number of references are given.

In Chapter I the author gives a brief outline of the theory of functions which depend on all the values of a function in a certain domain, or functions of lines as distinguished from ordinary functions of points. The expression

$$A = \int_a^b \varphi(x)dx$$

is a function of all the values which $\varphi(x)$ takes in the interval (a, b). The form of $\varphi(x)$ determines the variation of A or, in other words, A depends on all the ordinates of the curve $y = \varphi(x)$ in the interval (a, b) and is hence a function of an infinite number of variables. To indicate the fact that a quantity F depends on all the values of a function $\varphi(x)$ in an interval (a, b) the notation

$$F = F|[\varphi_a{}^b(x)]|$$

is used. Starting from this notion the author defines continuity, derivatives, etc., and gives several examples of functions of this kind. The treatment is very brief, and rigorous proofs are omitted, but numerous references are given for the benefit of the reader who wishes to penetrate deeper into this most fascinating field. Confining himself to functions which

are developable in a series corresponding to Taylor's series, the author is finally led to the consideration of a general functional equation of which the ordinary linear integral equation is a special case.

In Chapter II the author begins a systematic treatment of integral equations. He first considers two mechanical problems of Abel and Liouville, and then takes up a general discussion of the integral equation of the Volterra type of the second kind, i. e., an equation of the form

$$\varphi(x) = u(x) + \int_0^x K(x, \xi)u(\xi)d\xi,$$

where the kernel $K(x, \xi)$ is supposed to be finite and continuous in the region

$$0 \leqq \xi \leqq x \leqq b.$$

The integral equation is considered as a limiting case for a system of n algebraic equations as n becomes infinite.

The integral equation of the first kind, i. e.,

$$\varphi(x) = \int_0^x K(x, \xi)u(\xi)d\xi,$$

is considered next. Here both K and $\partial K/\partial x$ are supposed to be finite and continuous, and also $\varphi(0) = 0$. A solution is obtained by reducing the given equation to an equation of the second kind. The special case when K becomes infinite of an order $\alpha < 1$ for $x = \xi$ is then discussed. Finally the case when $K(x, x) = 0$ is considered. In discussing this case the elegant method of Lalesco, which consists in making use of linear differential equations, is applied.

The author next considers systems of integral equations and equations containing multiple integrals and shows how the ordinary methods can without serious complications be extended to these cases. A few pages are then devoted to the method of successive approximations as applied to integral equations. Other topics of which a brief discussion is given are generalized equations of the Volterra type, equations where the two limits of integration are variable, and non-linear equations.

In Chapter III equations of the Fredholm type

$$\varphi(x) = u(x) + \lambda \int_0^1 K(x, \xi)u(\xi)d\xi$$

are discussed. The same method of considering the given equation as the limiting case of a system of algebraic equations is used in treating this type. The kernel is supposed to be finite. The case where the kernel becomes infinite is discussed very briefly. A few pages are devoted to systems of equations and equations involving multiple integrals. At the end of the chapter some very interesting applications are made to Dirichlet's problem and to the vibration of strings.

In Chapter IV a very brief and incomplete account of integro-differential equations is given. By such an equation is meant one involving not only the unknown functions under signs of integration but also the unknown functions themselves and their derivatives. No attempt is made to give a systematic treatment of this subject. A few problems from mechanics leading to equations of this kind are discussed. A few pages are also devoted to permutable functions. For a more complete discussion of these very interesting topics the reader is referred to papers by Volterra published in *Acta Mathematica* and *Atti d. R. Accademia dei Lincei*. It is with great pleasure that we receive the news that the author intends to give an exhaustive treatment of these topics in a second volume which will soon be published.

JACOB WESTLUND.

Die komplexen Veränderlichen und ihre Funktionen. Von Dr. GERHARD KOWALEWSKI, Ord. Professor an der Hochschule zu Prag. Leipzig und Berlin, B. G. Teubner, 1911. 455 pp.

THIS volume, by the well-known author of the recently published text on determinants, is intended to be a continuation of the Grundzüge der Differential- und Integralrechnung, which was reviewed on page 531 of volume 19 of the BULLETIN, as well as an introduction to the theory of functions. Some of the very convenient terms introduced in the aforementioned book, such as the expression " fast alle," meaning " all with a finite number of exceptions," so useful in the discussion of propositions involving the limits of sequences, are also used in this book. Other new ones are introduced, as for instance the " Hof " of a point, meaning a circle having the given point as a center. This strikingly descriptive terminology, as well as the interjection in appropriate places

of interesting historical remarks,* lend to the style of this author a certain charm and clearness which it is difficult to characterize more specifically, but which must be evident to any one who reads his books; among these I should like to mention his Klassische Probleme der Analysis des Unendlichen, a book that seems to the present reviewer to be well worth the attention of teachers of the calculus. Professor Kowalewski succeeds in a remarkable way in bringing at least some features of the more advanced portions of analysis within the reach of the less advanced student; this gives his books added pedagogical value, inasmuch as they arouse the reader's interest in the regions lying a little beyond his present capacity.

The text on the complex variable and its functions possesses all the good qualities of the former productions of this author. Let me say at the outset that the book treats only of uniform functions, so that with the exception of a brief passage on page 319, no mention is made of a Riemann surface nor of any of the subjects connected therewith. The book is divided into seven chapters, the first one of which, comprising 64 pages, deals with the representation of complex numbers by means of the points of the plane, and with a discussion of the general linear fractional function of the complex variable z. In addition to the topics usually treated under this heading, the author gives in this chapter: first, a treatment of the transformation $z' = (a\bar{z} + b)/(c\bar{z} + d)$, where $\bar{z}$ is the conjugate of z; this section closes with a proof of the fact that all the elementary transformations, including rotations and translations, can be compounded out of reflections; second, a treatment of the linear fractional function in the homogeneous form, which gives an opportunity to bring in some interesting facts about Hermitian forms. The classification of finite linear transformation groups, including the modular group, leads to a digression on the equivalence of positive quadratic forms and to a short study of linear and planar "Punktgitter," useful later on in the discussion of doubly periodic functions.

Chapter II, 18 pages, is devoted to complex functions of a

* Such as, for instance, the remark on p. 91 of the present volume, just before a proof of the fundamental theorem of algebra, that Leibniz did not believe it to be possible to solve the equation $z^4 + 1 = 0$ by the aid of complex numbers.

real variable. The first and second mean-value theorems of the integral calculus are proved with the utmost rigor by means of the notion of the "smallest convex enclosure" of a given point set M. By this is meant the set consisting of all the points common to all the convex sets which contain M. It is proved that whenever M is closed and connected, then its smallest convex enclosure consists of all the points which lie on chords of M.

In Chapter III comes the introduction of the analytic or monogenic function, defined as a function for which the derivative exists. The definition of the derivative is given in a beautifully clear but at the same time completely rigorous way; the term "ausgezeichnete Folge," used so successfully in the "Grundzüge," and the theorem that all sub-sequences of a given convergent sequence are convergent, and that their limit is the same as that of the original sequence, applied to great advantage throughout the book, help materially in accomplishing this result. The analytic character of rational functions, the fundamental theorem of algebra, the unlimited differentiability and integrability of power series in the interior of their circles of convergence are proved in the first 26 pages of this chapter. The remaining 25 pages are given up to the exponential and trigonometric functions, defined by means of power series, and to the inversion of the former, the logarithmic function, closing with a section on functions of a function. The Mercator map is introduced by way of the conformal representation of a cylinder upon a circular ring. In the discussion of the logarithmic function, the different determinations are kept distinct and treated as different functions; it is shown that each one of them is determined by its derivative and by its value at one point. The expansions of the principal logarithms of $1+z$ and of $(1-z)/(1+z)$ lead very naturally to a discussion of series of the form

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos n\varphi}{n^p} \quad \text{and} \quad \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin n\varphi}{n^p} \qquad (p=1,2,\cdots).$$

Chapter IV, entitled "Curvilinear integrals," contains within its 46 pages propositions relating to the integrals of functions of a complex variable along rectifiable curves. The existence of

$$\int_{AB} f(z)dz$$

is proved for a continuous function $f(z)$ by reducing to integrals of functions of a real variable, and also directly without separating into real and imaginary parts. It would take us too far afield to report in detail upon the mode of presentation, which is a model of clearness and rigor. For the fundamental theorem that a function which has the derivative $f(z)$ at every point of a path AB is representable in the form

$$C + \int_{AB} f(z)dz,$$

the path of integration is further conditioned so as to have the following property: If $z_1, z_2, \cdots$ is a sequence of points on AB, converging to a point z_0 of AB, but not containing z_0, and if l_n is the length of the path z_0z_n, then $\lim l_n/|z_0 - z_n|$ must exist and be finite. This condition is equivalent to the one used by Moore in his proof of the Cauchy-Goursat theorem (see *Transactions of the American Mathematical Society*, volume 1), but seems to be simpler in form. All paths of integration considered from here on are assumed as having this property. Formulas are obtained for the coefficients of a power series in the usual way and also by means of Hadamard's formulas in terms of the real and imaginary parts of $f(z)$ expressed in terms of polar coordinates. The latter results are then used to derive Poisson's integral.

Chapter V is devoted to Cauchy's fundamental theorem and its consequences. It covers 143 pages and is the longest single chapter in the book. Cauchy's fundamental theorem is given first for a rectangle, by the method of Moore. A little farther on follow two proofs of the same fundamental theorem for so-called "Normalbereiche," defined as the set of points given by the formula

$$z = \{u + i(1-t)\psi_0(u) + it\psi_1(u)\}\, e^{i\gamma} \quad (a \leqq u \leqq b, 0 \leqq t \leqq 1),$$

where $\psi_0(u)$ and $\psi_1(u)$ are real and continuous functions of limited variation on the interval (ab), subject to the restriction $\psi_0(u) < \psi_1(u)$. The first of these proofs proceeds by means of decomposition into rectangles; the second one proceeds directly by a generalization of the method used for the

case of a rectangle. By the same method again, Green's theorem is proved for two functions $u(x, y)$ and $v(x, y)$, which are assumed to be "properly differentiable." A function $f(x, y)$ is said to be properly differentiable at a point (x_0, y_0), if there exist two functions $f_1(x, y)$ and $f_2(x, y)$ such that

$$\lim \frac{f(x_n, y_n) - f(x_0, y_0) - (x_n - x_0) f_1(x_0, y_0) - (y_n - y_0) f_2(x_0, y_0)}{|x_n - x_0| + |y_n - y_0|} = 0,$$

whenever $\lim x_n = x_0$ and $\lim y_n = y_0$, $(x_n, y_n \neq x_0, y_0)$.

From this it follows that the function $f(z)$ possesses a derivative at the point z_0, *when and only when* its real and imaginary parts u and v are properly differentiable at (x_0, y_0) and satisfy the conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

This result seems distinctly valuable inasmuch as it states the exact condition under which the two definitions of an analytic function are equivalent. Next come, very much as is customary, the important consequences of Cauchy's theorem: Cauchy's formula, Taylor's series, Laurent's series; followed by Liouville's theorem, the theorems on rational functions and residues, the second proof of the fundamental theorem, etc. The author proposes the term "Laurent point" of a function for a point at which the function is either regular or has an isolated singular point, i. e. a point at which the function may be developed into a Laurent series, a regular point then becoming a "regular Laurent point." The use of the proposed term throughout the rest of the book is a good argument in favor of its acceptance, for it enables the author frequently to simplify his statements. Between the Taylor series and the Laurent series, 15 pages are devoted to boundary-value problems. There is a clear treatment of the two-dimensional boundary-value problem for the circle and of the reduction of this problem and of the corresponding one for the ellipse to linear integral equations. This is one of those instances in which the author has brought the more advanced topics of his subject within the scope of a more elementary text. The last 34 pages of the chapter are devoted to the inversion of a system of two functions $u(x, y)$ and $v(x, y)$ and to the inversion of power series in the neighborhood of

simple and of multiple zeros of the function. The discussion of the latter case is the only place in the book where, as noticed above, a Riemann surface is mentioned. Conformal representation gets an admirable discussion in connection with this subject by means of the notion of a "Jordan neighborhood" of a point (x_0, y_0), by which is meant the totality of all points in the interior of a Jordan curve which surrounds the point (x_0, y_0). This chapter closes with the discussion of the transformation effected by $w = f(z)$ in the neighborhood of a multiple zero.

Chapters VI and VII take up the subjects of infinite series and infinite products of functions of a complex variable. The definitions and well-known theorems on uniform and absolute-uniform convergence are followed in section 68 by the definition of "normal convergence," as given by Baire in his Leçons sur les Théories générales de l'Analyse. Then comes the theorem that every uniformly convergent series can be transformed into a normally convergent series by the omission of a finite number of terms at the beginning of the series and by grouping of neighboring terms. No reference is given for this theorem; it seems to be due to Baire, who proves it on page 30 of volume 2 of the book, mentioned above. The complete absence from the text under review of references to the literature is unfortunate. Creating, as it cannot help doing, a great deal of interest in allied fields of mathematics, the book loses a great deal of its value by not giving the reader any guidance in his attempts to gain further knowledge of these fields. The theorem to which reference was made above and the concept of normal convergence, including as it does uniform and absolute-uniform convergence, prove to be very useful in several of the later proofs on infinite series and infinite products. The discussion on pages 348 and 349 of the derivatives of a uniformly convergent series of differentiable functions seems to be unnecessarily lengthened by the proof of the formulas for the successive derivatives of the function $s(\zeta)$; as soon as the existence of the first derivative is proved, these results can be obtained by application of the results of pages 215 and 216. The theorem of Heine-Borel (or rather that of Borel-Heine, as the author calls it with a regard for historical accuracy, not marred by national prejudice) is used to prove the following theorem: If P is a point set which contains interior points, if $f_n(z)$ are functions which are differenti-

able at every interior point of P, and if the series $\Sigma f_n(z)$ is uniformly convergent on every closed subset of P, which contains interior points only, then this series may be differentiated term by term indefinitely at every interior point and the resulting series converge uniformly on every such subset.

Nearly one half of these last two chapters, 64 pages, is given to doubly periodic functions and to elliptic integrals. The book ends with the presentation of the theorem of Mittag-Leffler and of Weierstrass's theorem on the factorization of entire functions. It is surprising that the subject of analytic continuation does not even receive mention.

The misprints are rather more numerous than is usual in Teubner's books, but they are not very important. It will suffice to point out the following:

Page 95: line 16 from top: read $R(z)$ instead of $Q(z)$.

Page 119: line 8 from bottom: read $F(\omega)$ instead of $G(\omega)$.

Page 129: line 6 from top: read $\frac{1}{2^3}+\frac{1}{4^3}+\frac{1}{6^3}+\cdots$ instead of $\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\cdots$.

Page 284: line 1 from bottom: read $\nu'+\nu''=\omega'+\omega''$ instead of: $\nu'=\nu''-\omega'+\omega''$

Page 349: lines 1 and 6 from top: the last exponent on each of these lines should read $-(p+2)$ instead of $-(p+1)$.

line 10 from top: a factor $M\rho$ should be multiplied into the second term on the right-hand side of the equation.

Page 375: line 3 from top: read 2π instead of $\pi/2$.

ARNOLD DRESDEN.

Calcul des Probabilités. Par LOUIS BACHELIER. Tome I. Paris, Gauthier-Villars, 1912. vii + 516 pp. Price 25 fr.

THE object of this book is to give not merely an exposition of some of the leading principles long known in the theory, but to present recent methods and results, due to the author, that represent from certain points of view a decided transformation of the calculus of probabilities.

The conception of continuous probabilities is at the foundation of this change. The author points out the fact that the

continuous formulas long used in the theory of probability have been thought of as approximations in such a way that they could not serve as a basis for new research. To this fact he attributes the failure to make greater progress in the development of the theory since the time of Laplace. This idea of considering probabilities as continuous is at the basis of much of the author's work published within the past ten or twelve years in the *Annales de l'Ecole Normale Supérieure* and *Comptes Rendus*. The book gives a unified presentation of the development of the conception of continuous probabilities contained in these papers, and shows some generalizations.

It is easy to see one advantage of continuous probabilities in that the results are mathematically exact and do not depend upon approximations. The theory has also made possible new results and fortunately some of these results are well adapted to numerical applications.

It should perhaps be said that the first five chapters of the book are given to discontinuous probabilities. This part contains nothing novel, and the effort has been toward simplification.

To consider briefly the method of treatment of continuous probabilities, let us assume the continuity of a function that represents a probability. To satisfy this condition we may consider a series of a great number of trials, of such nature that the succession of these trials is regarded as continuous, and that each trial may be regarded as an element. If we are concerned with a great number of trials, we may assume that they follow each other in small equal infinitesimal intervals of time dt, and represent the total time by t. This likeness between trials and time furnishes a valuable image which helps us to conceive of the transformation of probabilities in a series of trials as a continuous phenomenon.

In this book, probabilities are classified from three points of view. First, the classification is made with respect to conditions of play in the game with which we are concerned. If the conditions of play are identical all the time or for each μ, uniformity is said to exist. If conditions depend uniquely on the order in the series and are independent of what has gone before, there is said to be independence. If the conditions depend on what has happened before in the playing, there is said to be connexity (connexité). Second, the classification is made with respect to the number of players. Prob-

lems may involve one, two, three, . . ., or n players. Third, the classification is made with respect to the values of the variables, or the values of the fortunes of the players. When variables may take values from $-\infty$ to $+\infty$, the probabilities are said to be of the first kind; when one variable is limited in value in one direction, the probabilities are said to be of the second kind; when one variable is limited in both directions, the probabilities are said to be of the third kind; when two or several variables are limited, the probabilities are said to be of superior kind.

To illustrate the general character of the treatment, let us consider the case of probabilities of the first kind with one variable and under independence. Further, assume that a function $f(\mu_\alpha, \mu_\beta, y)$ exists such that $f(\mu_\alpha, \mu_\beta, y)dy$ is the probability of a loss between y and $y + dy$ in the playing of matches between μ_α and μ_β. Then, for the probability of a loss between x and $x + dx$ at the match μ, under the specified condition that at match μ_1 the loss is between x_1 and $x_1 + dx_1$, we have

$$f(0, \mu, x)dx = f(0, \mu_1, x_1) \cdot f(\mu_1, \mu, x - x_1)dx_1dx,$$

and, since x_1 may take values from $-\infty$ to $+\infty$, we have that the function f must satisfy the functional equation

$$f(0, \mu, x) = \int_{-\infty}^{+\infty} f(0, \mu_1, x_1)f(\mu_1, \mu, x - x_1)dx_1.$$

Furthermore, the condition of continuity requires that

$$\lim_{\mu = \mu_1} \int_{-a}^{b} f(\mu_1, \mu, y)dy = 1,$$

where a and b are any assigned positive numbers.

It is shown that these fundamental relations imply a certain partial differential equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{4\psi'(\mu)}{\varphi'(\mu)}\frac{\partial f}{\partial x} - \frac{4}{\varphi'(\mu)}\frac{\partial f}{\partial \mu} = 0,$$

where $\psi'(\mu)$ and $\varphi'(\mu)$ are known functions. In fact, $\psi'(\mu)d\mu$ is the mathematical hope of the interval μ to $\mu + d\mu$, and $\varphi'(\mu)d\mu$ is the function of instability for the same interval. In the case of symmetry of probabilities, the equation becomes

$$\frac{\partial^2 f}{\partial x^2} - \frac{4}{\varphi'(\mu)}\frac{\partial f}{\partial \mu} = 0.$$

If we make $\mu = t$, and think of it as the time, we have the familiar equation of Fourier, and we note that the theory of continuous probabilities has led to a likeness between what may be called the movement or transformation of probabilities and certain physical phenomena. This theory of probability serves logically as an introduction to mathematical physics, not only because a knowledge of the laws of chance often supplements our ignorance of the laws of nature, but also, because the theory, based on purely mathematical conceptions, leads to differential equations that are of fundamental importance in physics.

In the chapter on the radiation (rayonnement) of probabilities, we find established, under an assumption of uniformity, that a state (cours) radiates towards a neighboring state a quantity of probability proportional to the difference of their probabilities. This theorem and others on the radiation of probabilities struck me on first reading as involving a strained use of language. To explain the meaning: the probability that a state of gain or loss be x at time t and $x + \mu$ at time $t + dt$ is expressed by saying that the state x has in time dt given to state $x + \mu$ a quantity of probability equal to the probability of the combined states at the times specified. Considerable deliberation on the subject has led me to feel that this view of the radiation of probabilities is a rather natural conception; for, if a certain state has a high probability, we should naturally expect, under continuity, that this situation would tend to give to neighboring states increased probability.

The analogy of the above theorem to the theorem concerning the flow of heat from a body to a cooler body is discussed. This analogy no longer subsists in the problem either of connected probabilities or of independent probabilities of several variables. In these cases, the laws of probability are more complex than those of heat radiation.

In the illustrations that we have cited, independence is assumed. When one tries to free the development from this assumption, such difficulties are presented that the author considers only certain special classes of connected probabilities. He treats in elegant form connected probabilities of the first kind for one variable, by which it is meant that the conditions of play depend uniquely on the actual loss and on the order of the match. He treats in particular the simple but useful

case in which we conceive a cause accelerated or retarded by deviations proportional to the value of such deviations. To illustrate this problem, we may cite the following urn scheme: An urn A contains n white and n black balls, and a second urn B contains n white and n black balls. We draw at random a ball from A and place it in B, at the same time drawing one from B and placing it in A. If this process is continued μ times, what is the probability that the number of white balls in the urn is $n - x$? In this case, the mathematical hope of the player is x/n when the deviation is x.

The plan of extension from a single variable to any number of variables, in the case of independence, proceeds in a very direct and systematic manner. In fact, the unity of method in the development of the different classes of probabilities, founded on an integral equation, is one of the characteristic features of this work. Thus, if we let $f(\mu_1, \mu, x_1 - X_1, \cdots, x_{n-1} - X_{n-1})$ be the probability that, between matches μ_1 and μ, A loses $x_1 - X_1$, B loses $x_2 - X_2$, $\cdots$, the fundamental functional relation for independent probabilities of the first kind, is

$$f(\mu_0, \mu, x_1, \cdots, x_{n-1})$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(\mu_0, \mu_1, X_1, \cdots, X_{n-1}) \times f(\mu_1, \mu, x_1 - X_1, \cdots, x_{n-1} - X_{n-1}) dX_1 dX_2 \cdots dX_{n-1},$$

with the further condition that

$$\text{limit} \int_{-\alpha_1}^{\alpha_2} \int_{-\beta_1}^{\beta_2} \cdots \int_{-\gamma_1}^{\gamma_2} f(\mu_1, \mu, u_1, u_2, \cdots, u_{n-1}) \times du_1 du_2 \cdots du_{n-1} = 1 \begin{pmatrix} \alpha\text{'s and } \beta\text{'s} \\ \text{positive} \end{pmatrix}$$

for $\mu - \mu_1 \doteq 0$, and $u_1 = x_1 - X_1$, $u_2 = x_2 - X_2$, $\cdots$, $u_{n-1} = x_{n-1} - X_{n-1}$.

The function f is determined by these conditions. The analogy of the results with those of Pearson on multiple correlation is worth noting.

The book contains a short chapter on geometrical probabilities, one on kinematic probabilities, and one on dynamic probabilities. What is meant by kinematic probabilities may be expressed by saying that a problem has to do with kine-

matic probabilities when it requires a treatment of displacements that depend wholly or in part on chance. A problem of dynamic probabilities is one that has to do with movements of a system under forecs that depend wholly or in part on chance.

In conclusion, let me say that the fact that the book gives practically no references makes it difficult to determine just what is due to the author and what is derived from earlier authority. However, much of this work on continuous probability is original with the author, and it appears to the reviewer that this first volume gives a systematic and unified presentation of the author's contributions to the development of a conception of probability that makes possible a distinct advance in this field of mathematics, and in our notions of the application of probability theory. It is further a fact of some interest that integral equations play a fundamental part in this treatment of probability.

H. L. RIETZ.

Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation. I. *Physikalischer Teil.* Von A. EINSTEIN. II. *Mathematischer Teil.* Von M. GROSSMANN. Leipzig, B. G. Teubner, 1913. 38 pp.

EINSTEIN no sooner had defined the principle of relativity and established it on a sound basis than he went about destroying it, as some would say, or generalizing it, as he says, so as to take account of gravitational phenomena. A fundamental point of view in the original theory of relativity is that mass and energy are proportional; the new theory says that mass and weight are also proportional, for example, a ray of light is attracted by matter. The uniform rectilinear velocity of light in " free space " is therefore abandoned, or to put it differently, the presence of matter anywhere renders all space no longer free. The mathematical part of the theory will be especially interesting to those familiar with quadratic differential forms and Ricci's absolute calculus.

The pamphlet contains the most recent and detailed presentation of revised relativity; it is merely a reprint with repagination of an article in the *Zeitschrift für Mathematik und Physik*, volume 62.

E. B. WILSON.

NOTES.

THE opening (January) number of volume 15 of the *Transactions of the American Mathematical Society* contains the following papers: "On the degree of convergence of Laplace's series," by T. H. GRONWALL; "Linear associative algebras and abelian equations," by L. E. DICKSON; "Some theorems concerning groups whose orders are powers of a prime," by W. B. FITE; "Limits in terms of order, with example of limiting element not approachable by a sequence," by R. E. ROOT; "The symbolical theory of finite expansions," by O. E. GLENN; "Lebesgue integrals containing a parameter, with applications," by B. H. CAMP; "Congruences and complexes of circles," by J. L. COOLIDGE.

AT the meeting of the London mathematical society held December 12 the following papers were read: By E. W. HOBSON, "The linear integral equation"; by H. E. J. CURZON, "Generalized Hermite functions and their connection with Bessel functions"; by J. PROUDMAN, "Limiting forms of long period tides"; by A. L. CUNNINGHAM, "The number of primes of same residuality"; by R. H. FOWLER, "Some results on the form near infinity of real continuous solutions of a certain type of second order differential equations"; by S. BRODETSKY, "The potential of a homogeneous convex body and the direct integration of the potential of an ellipsoid"; by G. R. GODDSBROUGH, "The dynamical theory of the tides in a polar basin"; by J. C. FIELDS, "Proof of the complementary theorem."

AT the International conference on the teaching of mathematics, to be held at Paris, April 1–4, the first day will be devoted to the assembling of the delegates, meeting of the central committee, and an evening session of the Société mathématique de France. At the general session on Thursday there will be opening addresses by Professors P. APPELL and FELIX KLEIN, followed by lectures by Professor E. BOREL on "The adaptation of teaching to the progress of science," and Professor M. D'OCAGNE on "The rôle of mathematics in engineering sciences." The afternoon session will be devoted to the discussion of the introduction of the calculus and the use of elementary functions in secondary instruction. Reports

will be presented by Professors E. BEKE and C. BIOCHE. The Friday sessions will discuss the teaching of mathematics to engineering students, with a report by Professor P. STAECKEL. In the evening there will be a meeting of the Society of civil engineers of France. On Saturday the discussions will be continued and policies outlined in preparation for the meeting at Munich in 1915.

AT the annual public meeting of the academy of sciences of Paris, the permanent secretary, Professor GASTON DARBOUX, delivered an eulogy on HENRI POINCARÉ. The following prizes were awarded in mathematics, some of which had been announced earlier in the year: The Francoeur prize (2000 fr.) to A. CLAUDE for the whole of his work in astronomy. The Poncelet prize (3000 fr.) to MAURICE LEBLANC for his work in mechanics. The Petit d'Ormay prize (10000 fr.) to Professor C. GUICHARD for the whole of his work in mathematics. The Montyon prize (700 fr.) to Professor M. SAUVAGE for his work in mechanics. The Bordin, Gutzman, and grand prizes were not awarded.

PROFESSOR E. W. BROWN, of Yale University, has accepted the invitation of the British association for the advancement of science to attend the Australia meeting in 1914. He expects to be absent from the university until February, 1915.

PROFESSOR EDWARD KASNER, of Columbia University, has been granted leave of absence during the second term of the present academic year, to study in Europe.

DR. ELIZABETH B. COWLEY, of Vassar College, has been promoted to an assistant professorship of mathematics.

MR. A. L. MILLER has been appointed instructor in mathematics at the University of Michigan.

PROFESSOR BENJAMIN OSGOOD PEIRCE, of Harvard University, died January 14 at the age of 60 years. He was a member of the National academy of sciences and of many learned societies of Europe and America. He had been a member of the American Mathematical Society since 1891, and was vice-president of the Society in 1913.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BARBETTE (E.) Sur les carrés panmagiques. Paris, Gauthier-Villars, 1913. 8vo. 32 pp. Fr. 1.50

BATTAGLINI (G.). See TODHUNTER (I.).

BAUER (W.) und HANXLEDEN (E. v.). Differential- und Integralrechnung nebst den Grundzügen der synthetischen und darstellenden Geometrie. Braunschweig, Vieweg, 1913. 8vo. 10+216 pp. Cloth. M. 4.20

BETTI (E.). Opere matematiche, pubblicate per cura della r. accademia dei Lincei. Tomo II. Milano, Hoepli, 1913. 8vo. 8+496 pp.

BOLYAI (W.) und BOLYAI (J.). Geometrische Untersuchungen. Mit Unterstützung der ungarischen Akademie der Wissenschaften herausgegeben von P. Stäckel. 2 Teile. (Urkunden zur Geschichte der nichteuklidischen Geometrie. Band II.) Leipzig, Teubner, 1913. 8vo. 293 and 281 pp. Cloth. M. 32.00

BOUTROUX (P.). Les principes de l'analyse mathématique, exposé historique. Tome 1er. Paris, Hermann, 1914. 8vo. 11+547 pp. Fr. 14.00

CAHEN (E. P.). Théorie des nombres. Tome 1er: Le premier degré. Paris, Hermann, 1914. 8vo. 12+408 pp. Fr. 14.00

CANTOR (M.). Vorlesungen über Geschichte der Mathematik. 2ter Band: Von 1200–1668. Unveränderter Neudruck der 2ten Auflage. Leipzig, Teubner, 1913. 8vo. 12+943 pp. M. 29.00

CREMONA (L.). Opere matematiche, pubblicate sotto gli auspici della r. accademia dei Lincei. Tomo I. Milano, Hoepli, 1914. 4to. 8+497 pp. L. 25.00

DARBOUX (G.). Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal. 1re partie: Généralités, coordonnées curvilignes; surfaces minima. 2e édition, revue et augmentée. Paris, Gauthier-Villars, 1914. 8vo. 7+618 pp. Fr. 20.00

ENCYCLOPÉDIE des sciences mathématiques. Edition française. Tome II, 6e volume, 1re fascicule: Calcul des variations. Compléments. Par A. Kneser, E. Zermelo, H. Halm, et M. Lecat. Leipzig, Teubner, 1913. 8vo. Pp. 1–128. M. 4.80

FABRY (E.). Démonstration du théorème de Fermat. Paris, Hermann; 1913. 8vo. 23 pp. Fr. 1.50

FITZ-PATRICK (J.). Exercices d'arithmétique; énoncés et solutions. Avec une préface de J. Tannery. 3e édition entièrement refondue et considérablement augmentée. Paris, Hermann, 1914. 8vo. 5+707 pp. Fr. 12.00

GEBHARDT (M.). See WITTING (A.).

HANXLEDEN (E. v.). See BAUER (W.).

HEIBERG (J. L.). See TANNERY (P.).

HILBERT (D.). Théorie des corps de nombres algébriques. Ouvrage traduit de l'allemand par MM. Got et Lévy. Avec préface de G. Humbert. Paris, Hermann, 1912. 4to. 400 pp. Fr. 25.00

JUNK (W.). Rara historico-naturalia et mathematica. Volumen I (21 partes). Berlin, Junk, 1900–13. 8vo. 3+121 pp. M. 20.00

JUNKER (F.). Analysis (Differential- und Integralrechnung). Stuttgart, Ulmer, 1913. 8vo. 8+216 pp. Cloth. M. 2.40

KNOPP (K.). Funktionentheorie. 2ter Teil: Anwendungen der Theorie und Untersuchung spezieller analytischer Funktionen. (Sammlung Göschen. Nr. 703.) Berlin, Göschen, 1913. 8vo. 116 pp. Cloth. M. 0.90

LORIA (G.). Le scienze esatte nell'antica Grecia. 2a edizione totalmente riveduta. Milano, Hoepli, 1914. 16mo. 24+969 pp. L. 9.50

MAENNCHEN (P.). Geheimnisse der Rechenkünstler. (Mathematische Bibliothek. Nr. 13.) Leipzig, Teubner, 1913. 8vo. 4+48 pp. M. 0.80

MAIRE (A.). L'oeuvres scientifiques de Blaise Pascal. Bibliographie critique et analyse de tous les travaux qui s'y rapportent. Préface par P. Duhem. Paris, Hermann, 1913. 8vo. 30+184 pp. Fr. 15.00

PASCAL (BLAISE). See MAIRE (A.).

PLATRIER (C.). Sur les mineurs de la fonction déterminante de Fredholm et sur les équations intégrales linéaires. (Thèse.) Paris, Gauthier-Villars, 1913. 4to. 74 pp. Fr. 4.00

POINCARÉ (H.) L'oeuvres d'Henri Poincaré. Par Brunschvicg, Hadamard, Lebeuf, Langevin. (*Revue de Métaphysique et de Morale.*) Paris, Hermann, 1913. 8vo. 120 pp. Fr. 4.00

RIESZ (F.). Les systèmes d'équations linéaires à une infinité d'inconnues. (Collection de monographies sur la théorie des fonctions publiées sous la direction de M. Émile Borel.) Paris, Gauthier-Villars, 1913, 8vo. 6+182 pp. Fr. 6.50

SCHEFFERS (G.). Anwendung der Differential- und Integralrechnung auf Geometrie. 2ter Band: Einführung in die Theorie der Flächen. 2te verbesserte und vermehrte Auflage. Leipzig, Veit, 1913. 8vo. 11+582 pp. Cloth. M. 16.00

SCHOENFLIES (A.). Entwicklung der Mengenlehre und ihrer Anwendungen. Umarbeitung des im 8ten Bande der Jahresberichte der deutschen Mathematiker-Vereinigung erstatteten Berichts. Gemeinsam mit H. Hahn herausgegeben. 1te Hälfte: Allgemeine Theorie der unendlichen Mengen und Theorie der Punktmengen. Leipzig, Teubner, 1913. 8vo. 11+389 pp. M. 18.00

STÄCKEL (P.). See BOLYAI (W.).

SUAREZ (R.). Zum grossen Fermatschen Satz! Beweis. Dresden, Köhler, 1913. 8vo. 4 pp. M. 0.50

TANNERY (P.) Mémoires scientifiques, publiés par J. L. Heiberg et G. H. Zeuthen. Tome II: Sciences exactes dans l'antiquité. (1883–1898.) Paris, Gauthier-Villars, 1912. 8vo. 12+555 pp. Fr. 15.00

TODHUNTER (I.). Trattato sul calcolo. Versione dall'inglese, con aggiunte, di G. Battaglini. 5a edizione, riveduta ed aumentata sull'ultima edizione inglese. 2 volumi. Napoli, Pellerano, 1913. 8vo. 476 and 451 pp. L. 12.00

WEITZEL (C. G.). Unterrichtsbriefe zur Einführung in die höhere Mathematik. 1ter Band: Stereometrie und Trigonometrie. 8+472 pp.

2ter Band: Analysis, analytische Geometrie, Differential- und Integralrechnung. 8+480 pp. Wien, Hartleben, 1913. 8vo. M. 17.50

WHITE (C. E.). Theory of the irreducible cases of equations and its applications in algebra, geometry and trigonometry. Part II. Buckhannon, W. Va., White, 1913. 8vo. 4+90 pp.

WITTING (A.) und GEBHARDT (M.). Beispiele zur Geschichte der Mathematik. Ein mathematisch-historisches Lesebuch. 2ter Teil. (Mathematische Bibliothek. Heft 15.) Leipzig, Teubner, 1913. 8vo. 8+61 pp. M. 0.80

ZEUTHEN (G. H.). See TANNERY (P.).

ZORETTI (L.). Leçons de mathématiques générales. Avec une préface de P. Appell. Paris, Gauthier-Villars, 1913. 8vo. 16+754 pp. Cartonné. Fr. 20.00

II. ELEMENTARY MATHEMATICS.

BOREL (E.). Die Elemente der Mathematik. Deutsch von P. Stäckel. 1tes Heft: Aufgaben aus der Arithmetik und Algebra. Lösungen. Leipzig, Teubner, 1913. 8vo. 4+44 pp. M. 1.50

BOUCHENY (G.) et GUÉRINET (A.). La géométrie au cours complémentaire. Paris, Larousse, 1913. 8vo. 256 pp. Fr. 3.50

BRENKE (W. C.). See LONG (E.).

CAMMAN (P.) et FASSBINDER (C.). Algèbre et géométrie. Classe de 2e A et B. 2e édition revue et corrigée. Paris, Gigord, 1913. 12mo. 156 pp. Fr. 1.50

CAMMAN (P.) et GRIGNON (A.). Algèbre. Classes de 3e B, 2e et 1re C et D. 3e édition. Paris, Gigord, 1913. 8vo. 6+228 pp. Fr. 3.00

COMBEROUSSE (C. DE). Cours de mathématiques. 5e édition revue et augmentée. Tome 2, 2e partie: Trigonométrie rectiligne et sphérique. Paris, Gauthier-Villars, 1913. 8vo. Pp. 551–881. Fr. 5.00

COULOM (F.) et WEILL (M.). Nouveau cours de géométrie théorique et pratique. 1re année. Paris, Delagrave, 1913. 183 pp. Fr. 1.75

ELÉMENTS d'algèbre. Par F. I. C. Tours, Mame, 1913. 12mo. 8+409 pp.

EXERCICES et problèmes de géométrie. Par une reunion de professeurs. Paris, Gigord, 1913. 12mo. 96 pp.

FASSBINDER (C.). See CAMMAN (P.).

FAZZARI (G.). Elementi di aritmetica con note storiche e numerose questioni varie per le scuole superiori. Parte 1a. Palermo, Reber, 1911. 16mo. 8+128 pp. L. 1.40

GRIGNON (A.). See CAMMAN (P.).

GRÜNERT (A.). Tafeln zur Berechnung der Koordinaten von Polygonen- und Kleinpunkten. Stuttgart, Wittwer, 1914. 8vo. 7+181 pp. Cloth. M. 8.50

GUÉRINET (A.). See BOUCHENY (G.).

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